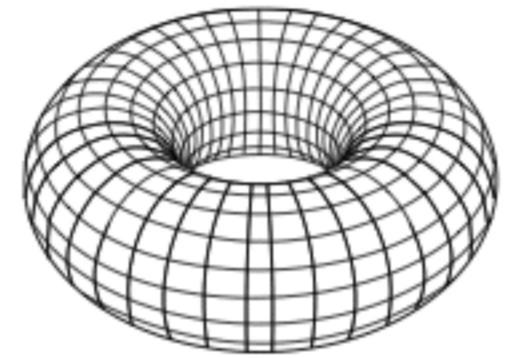
Manifolds & charts



A manifold **M** of dimension n is a topological space that near each point resembles n-dimensional Euclidean space $\mathbb{R}^n$. That is, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension n

Let $\mathcal{U} \subseteq \mathbf{M}$, and $\mathcal{V} \subseteq \mathbb{R}^n$.
A homeomorphism $\phi : \mathcal{U} \to \mathcal{V},\ \phi(u) = (x_1(u), \ldots, x_n(u))$
is a coordinate system on $\mathcal{U}$, and coordinates $x_1, \ldots, x_n$.
The pair $(\mathcal{U}, \phi)$ is called a coordinate chart

Equivalence principle: choice of charts is arbitrary

- **Space-time** is a **Lorentzian Manifold**

smooth manifold: locally similar enough to Euclidean space to do calculus

Pseudo-Riemannian manifold: equipped with an inner product on tangent space. Family of inner products is called Riemannian metric (tensor). Metric need not to be positive definite

Lorentzian manifold: 4 dimensional, (-,+,+,+)

→ tangent vectors can be classified into time-like, null, space-like

Some basic rules

- In curved space-times we cannot use Cartesian coordinates
- Expressions have to be coordinate invariant
- coordinate transformation for tensors

$$\tilde{g}_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\mu'}} \frac{\partial x^{\nu}}{\partial \tilde{x}^{\nu'}} g_{\mu\nu}$$

"covariant index"

or

$$\tilde{g}^{\mu'\nu'} = \frac{\partial \tilde{x}^{\mu'}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\nu'}}{\partial x^{\nu}} g^{\mu\nu}$$

"contravariant index"

$$S^{\alpha\beta}{}_{\gamma\cdots}{}^{\delta\cdots} = \cdots$$

- $g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma}$, i.e., $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$

The metric

$$\tilde{g}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial \tilde{x}^{\mu'}} \frac{\partial x^\nu}{\partial \tilde{x}^{\nu'}} g_{\mu\nu}$$

- In GR the most important tensor is the metric $g_{\mu\nu}$ which is the generalisation of $\eta_{\mu\nu}$
- line element $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$
- there exists a single point on the manifold such that $g_{\mu\nu} \to \eta_{\mu\nu}$ and even $\frac{\partial}{\partial x^\kappa} g_{\mu\nu} \to 0$ (local inertial frame, LIF)
- $g_{\mu\nu}$ is determined by Einstein's equations
- scalar product $g_{\mu\nu} a^\mu b^\nu = a_\nu b^\nu \equiv (a \cdot b)$

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Derivatives

- In Euclidean space: - $\mathrm{d}A_i$ of vector A_i is also a vector

  space-independent

  - $\partial_k A_i$ form a tensor

- This is generally **not** the case in curvilinear coordinates:

  - covariant vector $$A_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A'_\nu$$

  $x'^\nu = f'^\nu(x^\mu)$

  - its differential $$\mathrm{d}A_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\mathrm{d}A'_\nu + A'_\nu \frac{\partial^2 x'^\nu}{\partial x^\mu \partial x^\kappa}\mathrm{d}x^\kappa$$

- Must generalise partial derivative to covariant derivative

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The covariant derivative ∇_μ

- Cannot be "too different" from the partial derivative.
  Locally free falling coordinate system (LIF): $\nabla_\mu \to \partial_\mu$

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

- $\Gamma^\nu_{\mu\lambda}$ is the connection coefficient; result of parallel transport
  Locally free falling coordinate system (LIF): $\Gamma^\nu_{\mu\lambda} \to 0$

- $\xrightarrow{\text{LIF}}$ $\partial_\mu g_{\alpha\beta}(P) \equiv \nabla_\mu g_{\alpha\beta} = 0$ **Must also hold in any coordinate system!**

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu} \right)$$

Christoffel symbol $(\Gamma^\nu_{\mu\lambda} = \Gamma^\nu_{\lambda\nu})$

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Covariant derivatives II

- $\Gamma^{\alpha}_{\mu\nu}$ are constructed to be non-tensorial:
  Instead, $\nabla_\mu v^\alpha$ transforms as a tensor
- linearity $\nabla(T+S) = \nabla T + \nabla S$
- product $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$
- On scalars we simply have $\nabla_\mu \Phi = \partial_\mu \Phi$
- On vectors
$$\nabla_\mu v^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha_{\mu\nu} v^\nu$$
$$\nabla_\mu v_\alpha = \partial_\mu v_\alpha - \Gamma^\nu_{\alpha\mu} v_\nu$$