PhD Core Course on General Relativity

LECTURE 2

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Outline of the Course

- [x] Special Relativity and flat space-time
- [x] Manifolds
- [ ] Curvature & Gravitation
- [ ] Schwarzschild solution, black holes
- [ ] ADM formalism
- [ ] “Weyl" formalism of GR
- [ ] Cosmological perturbations: standard, gradient expansion, …
- [ ] Eulerian/Lagrangian fluid descriptions in GR

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So far we discussed

- Special Relativity, based on two key assumptions
  - Speed of light *c* is invariant in all reference frames
  - Law of physics are invariant in all inertial frames
- necessity to generalise to accelerated frames. Assume
  - equivalence of inertial & gravitational mass (WEP)
  - In a freely falling reference frame, Special Relativity must hold (Einstein's equivalence principle)
  - Gravitational field is described by the Riemann tensor

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So far we discussed II

- To do calculus in curvilinear coordinates, we need some basic tools:
  - ✦ line element: $$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$$
  - ✦ coordinate trafo for tensors: $$\tilde{g}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial \tilde{x}^{\mu'}} \frac{\partial x^\nu}{\partial \tilde{x}^{\nu'}} g_{\mu\nu}$$
  - ✦ need the covariant derivative to preserve tensorial character

$$\nabla_\mu v^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha_{\mu\nu} v^\nu$$

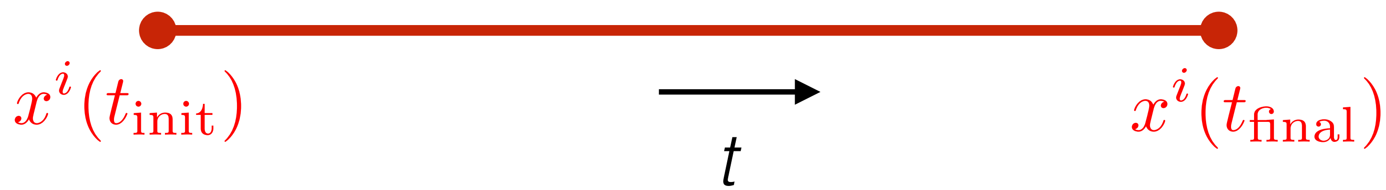
$$\nabla_\mu v_\alpha = \partial_\mu v_\alpha - \Gamma^\nu_{\alpha\mu} v_\nu$$

$$\nabla_\mu g_{\alpha\beta} = 0$$

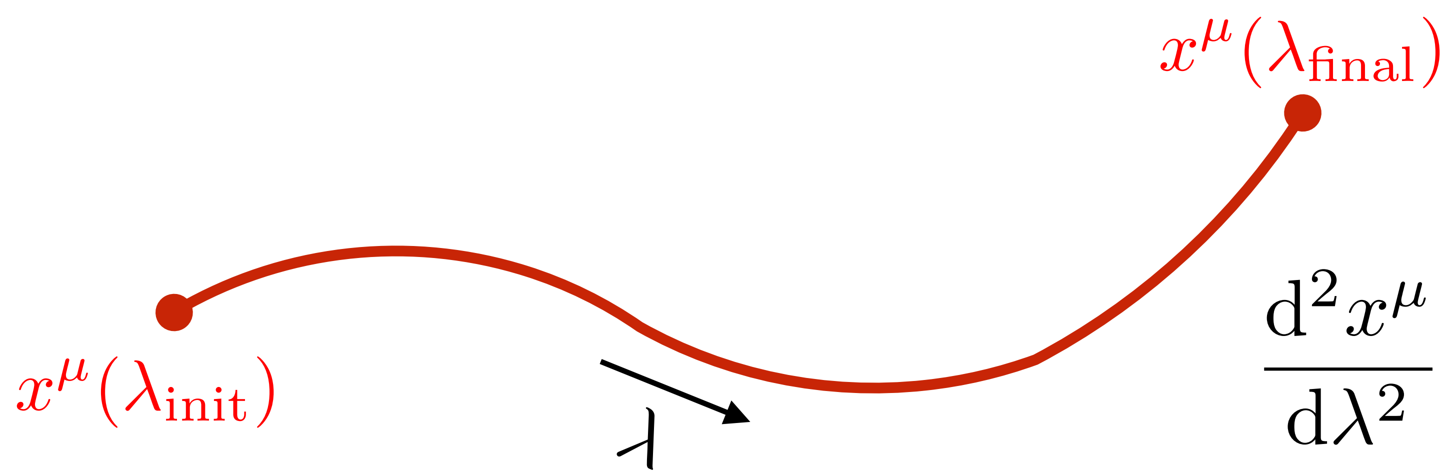
Christoffel symbol follows from

Geodesic equation

- Generalisation of Newton's law with no forces: $$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = 0$$



- in GR:



$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0$$

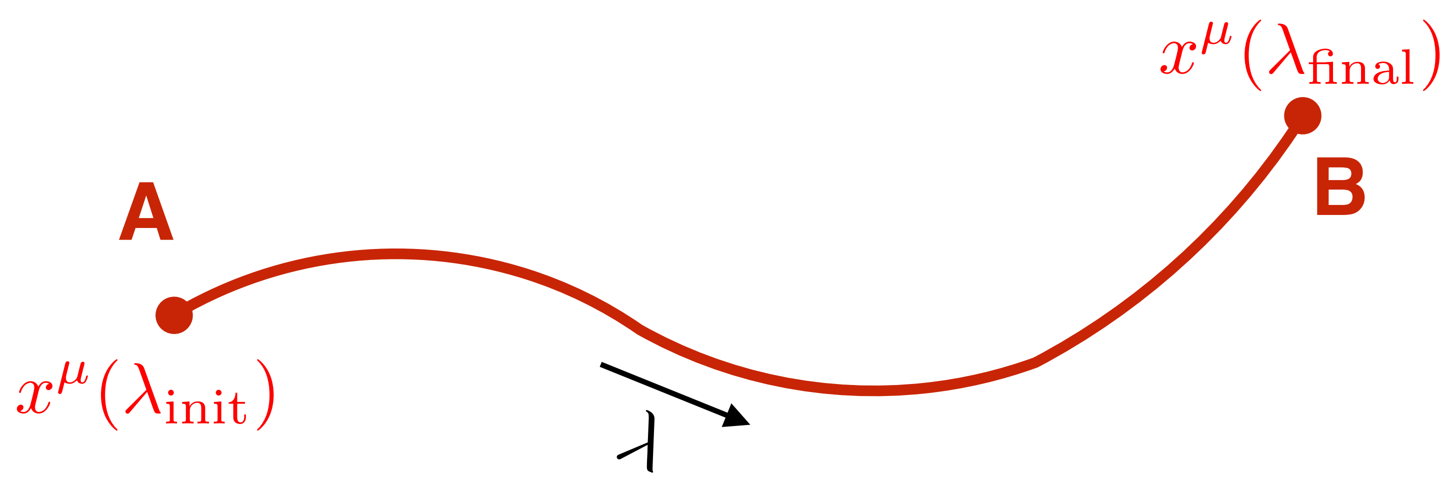
λ is an affine parameter
(can't use time in general since it's now a coordinate)

More about geodesics

- Geodesic equation can be derived from an action principle

- test-bodies follow paths that extremize the proper interval $\mathrm{d}s$

$$s_{\mathrm{AB}} = \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}s = \int_{\mathrm{A}}^{\mathrm{B}} \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \mathrm{d}\lambda$$



- Massless bodies move along null geodesics $(\mathrm{d}s^2 = 0)$
  Massive bodies move along time-like geodesics $(\mathrm{d}s^2 < 0)$

Riemann tensor

- Curvature is described by the Riemann tensor

$$R^{\mu}{}_{\alpha\beta\gamma} = \partial_{\beta}\Gamma^{\mu}_{\alpha\gamma} - \partial_{\gamma}\Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\beta\delta}\Gamma^{\delta}_{\alpha\gamma} - \Gamma^{\mu}_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta}$$

  Indices are lowered and raised with the metric

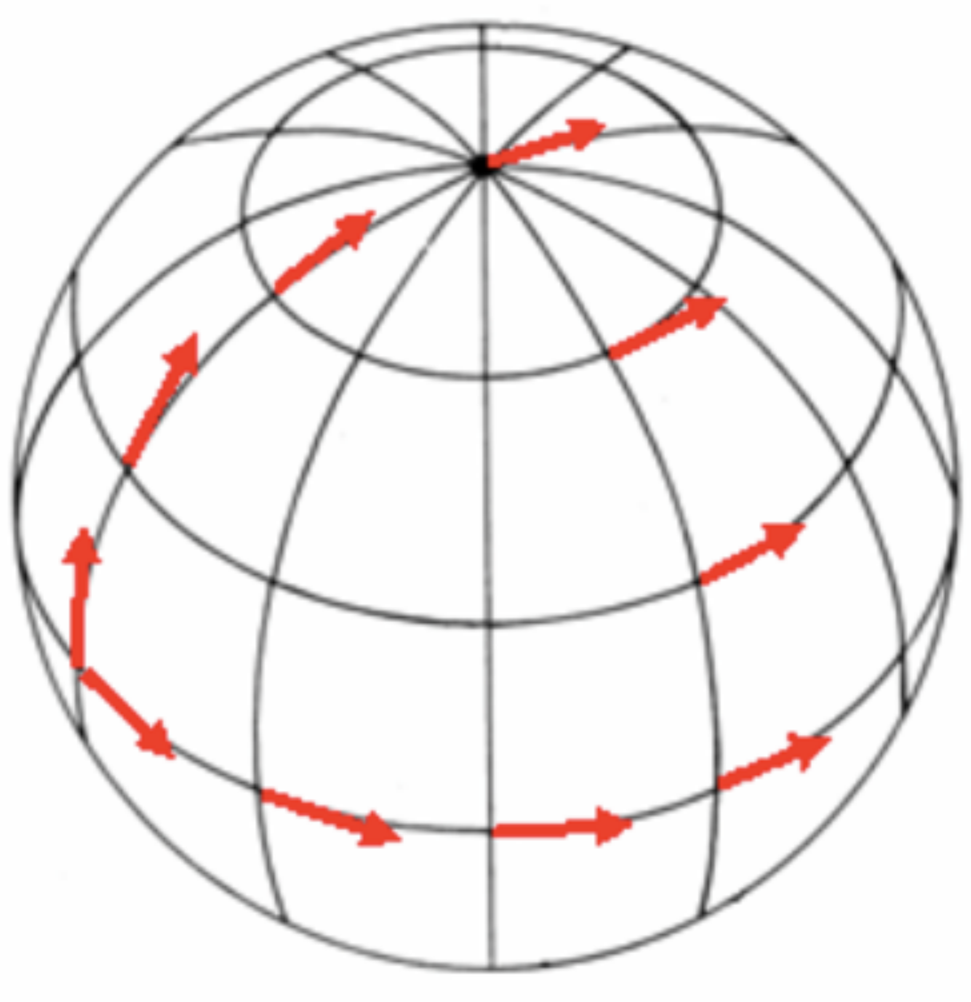
- Contains the information about the gravitational field (tidal force experienced by a rigid body moving along a geodesic)

- Is fully determined by the metric

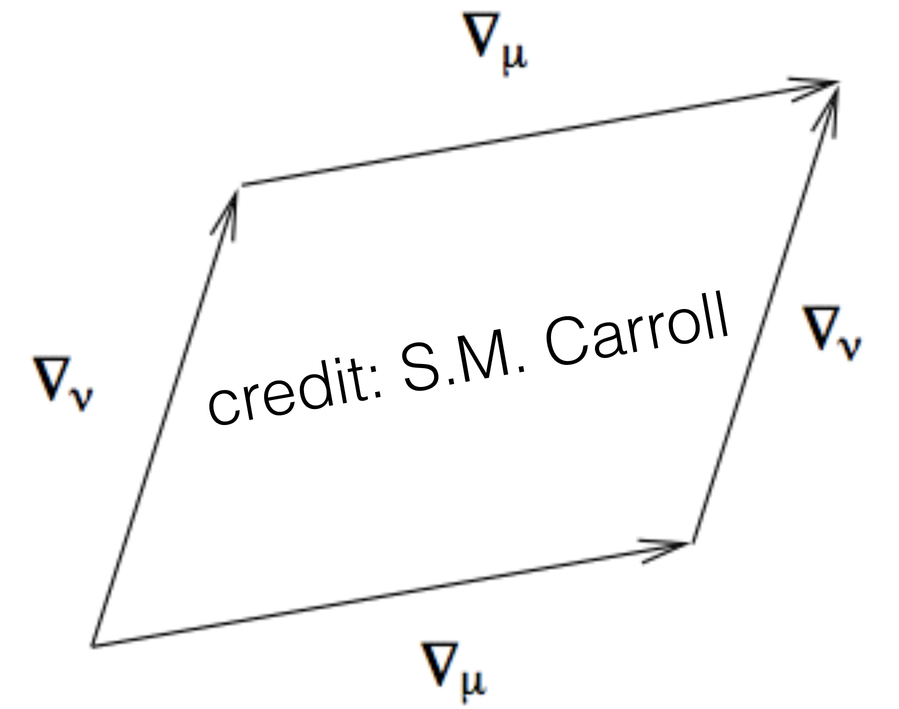
- Only vanishes when space-time is flat

Riemann tensor II

Interpretation: in curved space a vector parallel transported around a loop does not point in the original direction. The Riemann tensor measures this difference



Riemann tensor II



Technically:

Riemann tensor can be derived from the non-commutativity of covariant derivatives acting on an arbitrary vector

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho \\
&= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma}\nabla_\nu V^\sigma - (\mu \leftrightarrow \nu) \\
&= \partial_\mu\partial_\nu V^\rho + (\partial_\mu\Gamma^\rho_{\nu\sigma})V^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma}V^\sigma \\
&\qquad + \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}V^\lambda - (\mu \leftrightarrow \nu) \\
&= \underbrace{(\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})}_{\equiv R^\rho{}_{\sigma\mu\nu}}V^\sigma - 2\underbrace{\Gamma^\lambda_{[\mu\nu]}}_{\text{torsion} \to 0}\nabla_\lambda V^\rho\ .
\end{aligned}
$$

Riemann tensor III

- Properties:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$$ First Bianchi identity

$$\nabla_\delta R^\kappa{}_{\alpha\beta\gamma} + \nabla_\gamma R^\kappa{}_{\alpha\delta\beta} + \nabla_\beta R^\kappa{}_{\alpha\gamma\delta} = 0$$ Second Bianchi identity

- Symmetries: 256 -> 20 independent components

- comparison: metric has 10 components

Derived quantities

curvature generated by the matter fields that are present at any location

- Ricci tensor: $$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

- Ricci scalar: $$R = g^{\mu\nu} R_{\mu\nu}$$
- Einstein tensor: $$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$
- From the Bianchi identity: $$\nabla^{\mu} G_{\mu\nu} = 0$$
- Weyl/conformal tensor: (trace-free part of the Riemann tensor)

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\left(R_{\alpha\delta} g_{\beta\gamma} + R_{\beta\gamma} g_{\alpha\delta} - R_{\alpha\gamma} g_{\beta\delta} - R_{\beta\delta} g_{\alpha\gamma}\right) + \frac{1}{6} R \left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\right)$$

contains information about (free) gravitational radiation

10 independent components

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Possibilities to determine $g_{\mu\nu}$

- Need a tensor equation which constrains the 10 components of the metric

- Riemann tensor contains 20 components

- Option A: Use Einstein / Ricci tensor
  -> Einstein field equations

- Option B: Use Weyl tensor
  - > Maxwell-like equations

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