Gravitation

ICG
Portsmouth

Field equations

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}}$$

set now c=1

- LHS: curvature of space-time;
  RHS: energy-momentum of matter fields

- flat space: $\partial^\mu T_{\mu\nu} = 0$ curved space: $\nabla^\mu T_{\mu\nu} = 0$
  only **local** conservation in GR (Noether: time, translation)

- For a perfect fluid: $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$

fluid density

4-velocity

$g^{\mu\nu}U_\mu U_\nu = -1$

pressure

Newtonian limit?

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})$$

- factor κ is the result of matching the Newtonian limit

$$\begin{aligned} g_{00} &= -1 + h_{00} \\ g^{00} &= -1 - h_{00} \end{aligned} \longrightarrow T = g^{00}T_{00} = -T_{00}$$

$$R_{00} = \cdots = -\frac{1}{2}\nabla^2 h_{00}$$

- Zeroth component of field equations is then

$$R_{00} = \frac{1}{2}\kappa T_{00} \longrightarrow \boxed{\nabla^2 h_{00} = -\kappa T_{00}}$$

$$\updownarrow \qquad \updownarrow$$

comparison with Poisson equation $\nabla^2\Phi = 4\pi G\rho$

Cosmological example: Friedmann equations

- Start with $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$

cosmological constant

in the fluid's rest frame: $T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p)$

- Castrate metric to $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right]$

we want to solve for the cosmological factor

global 3-curvature: -1,0,1

- 00 component →

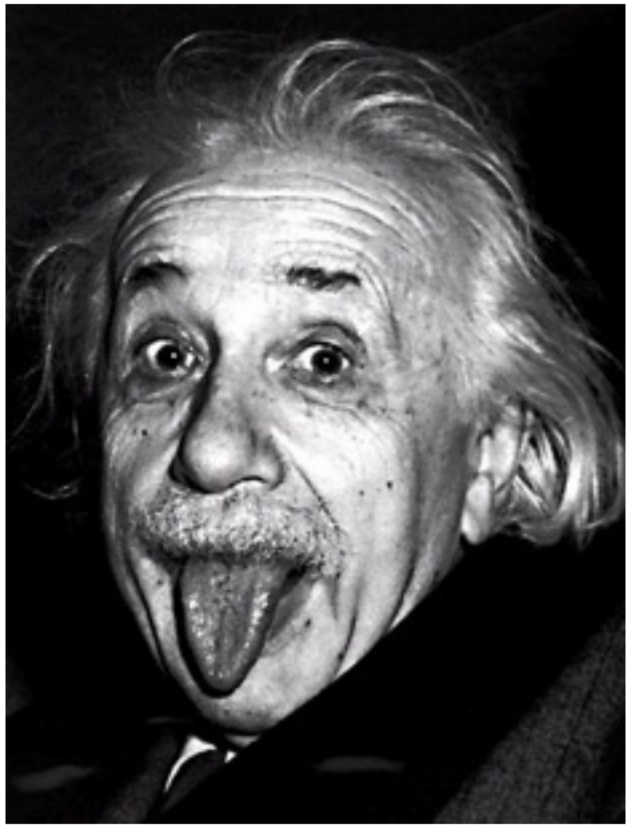
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}$$

- 00 + trace →

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

General structure of GR equations

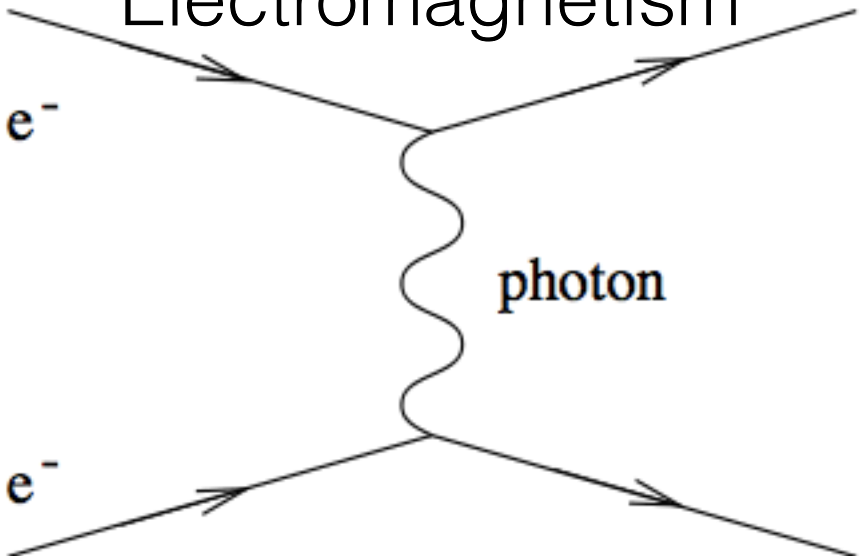
COMPLICATED!



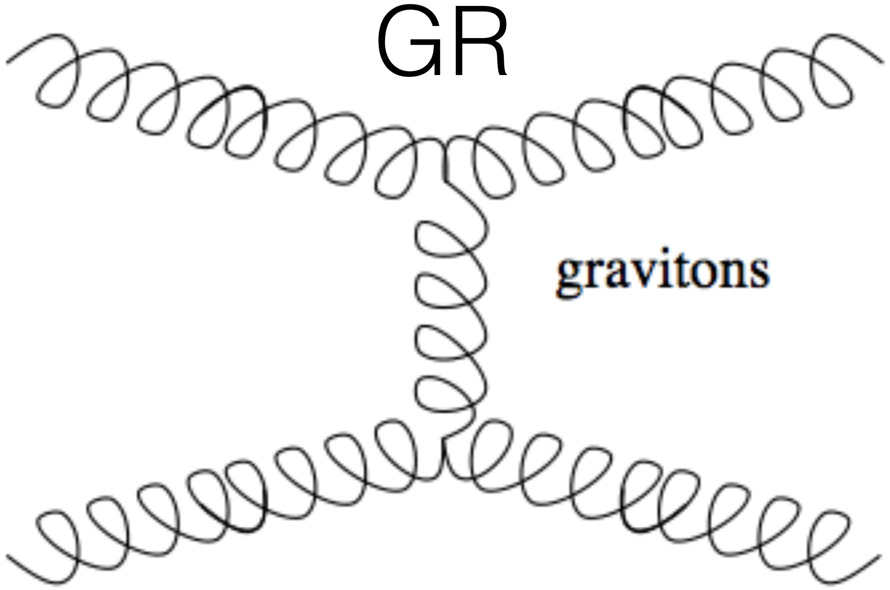
General structure of GR

- highly non-linear coupled set of differential equations
- even vacuum solution is tricky $R_{\mu\nu} = 0$
- in general, exact solutions are only known for very specific metrics (e.g., FLRW, Schwarzschild, Kerr)
- Gravitational field couples to itself, e.g.

Electromagnetism



GR



Some properties of the field equations

- Was first derived from the variation of the (Einstein-) Hilbert action

$$S = \int \mathrm{d}^4 x \sqrt{-g} R \longrightarrow \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

- include matter $T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$ and/or 2Λ , …
- $\nabla^\mu T_{\mu\nu} = 0 = \nabla^\mu G_{\mu\nu} \longrightarrow$ reduces the number of independent components to 10-4 =6 ⟵ **arbitrary choice of coordinates**
- Cauchy problem, use e.g. ADM formulation of GR

Schwarzschild solution

- describes spherically symmetric vacuum space-times (e.g., black holes, particle orbits in Schwarzschild geometry)
- fully non-linear solution to $R_{\mu\nu} = 0$
- Schwarzschild solution is the *unique* spherically symmetric solution (Birkhoff's theorem)

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2$$

- True singularity r=0 and a coordinate singularity $r = 2GM$
- becomes Minkowski for $M \to 0$ or $r \to \infty$ (asymptotic flat)

Real or coordinate singularities?

- Sufficient condition for a real singularity at point P:
  When one or more of the scalars of $R_{\alpha\beta\gamma\delta}$ blows up at P
- Coordinate singularities: study Schwarzschild metric and find more convenient coordinates. E.g., **tortoise coordinates**:

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \longrightarrow ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r^{*2}\right) + r^2 d\Omega^2$$

$$r^* \to -\infty \; (r \to 2GM)$$

asymptotic flat (large r)

some coordinates for black holes

- **tortoise coordinates**:

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \longrightarrow ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r^{*2}\right) + r^2 d\Omega^2$$

- Study radial null-geodesics: Eddington-Finkelstein

ingoing $v = t + r^*$, outgoing $u = t - r^*$ $\longrightarrow$ $\mathrm{d}s^2 = \cdots$

- Coordinate system with proper time $\bar{t}$ and proper distance $\bar{r}$ as measured from **a free-falling observer (Kruskal frame)**

$$\bar{v} = \bar{t} + \bar{r}\,, \quad \bar{u} = \bar{t} - \bar{r}$$

$$v = 4GM \ln \frac{\bar{v}}{4GM}\,, \quad u = -4GM \ln\left(-\frac{\bar{u}}{4GM}\right) \longrightarrow \mathrm{d}s^2 = \frac{2GM}{r(\bar{v},\bar{u})} \exp\left(1 - \frac{r(\bar{v},\bar{u})}{2GM}\right) \mathrm{d}\bar{v}\mathrm{d}\bar{u}$$