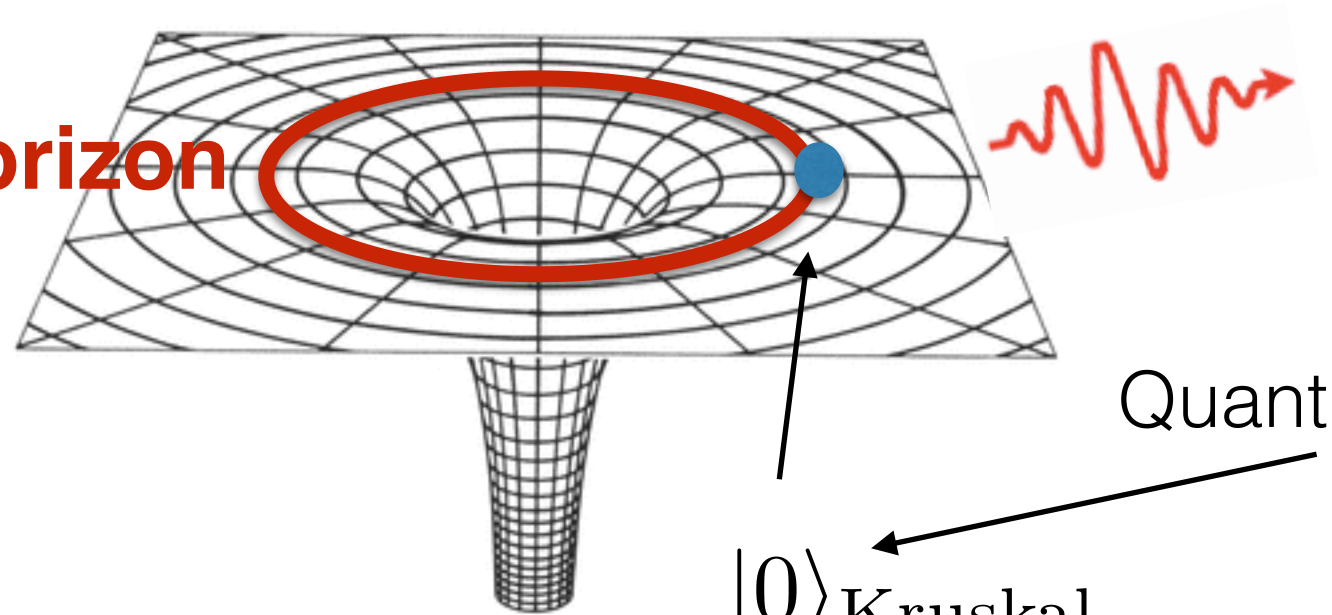
Hawking radiation

stationary observer

$|0\rangle_{\rm Tortoise}$

event horizon



Quantum vacuum states

$|0\rangle_{\rm Kruskal}$

different vacua

$\hat{a}^-|0\rangle_{\rm Kruskal} = 0 \quad \longleftrightarrow \quad \hat{b}^-|0\rangle_{\rm Tortoise} = 0$

Perform a Bogolyubov transformation of the creation/annihilation operators

e.g.: $\hat{b}^- = \alpha\hat{a}^- + \beta\hat{a}^\dagger$

This is a **coordinate transformation** of the quantum states

$$\longrightarrow \quad T_{\rm Hawking} = \frac{1}{8\pi GM}$$

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ADM formulation of GR

Or: how to obtain a Hamiltonian formulation

Or: how to obtain a Hamiltonian-Jacobi formulation

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Hamiltonian formulation of GR

- Consider Einstein-Hilbert action, coupled to CDM and Λ

cosmological constant

$$\mathcal{S} = \int \mathrm{d}^4 q \frac{\sqrt{-g}}{2} \left[{}^{(4)}R - 2\Lambda - \rho \left(g^{\mu\nu} \partial_\mu \partial_\nu \chi + 1 \right) \right]$$

CDM fluid density

$g = \det[g_{\mu\nu}]$, χ is the 4–velocity potential of the CDM

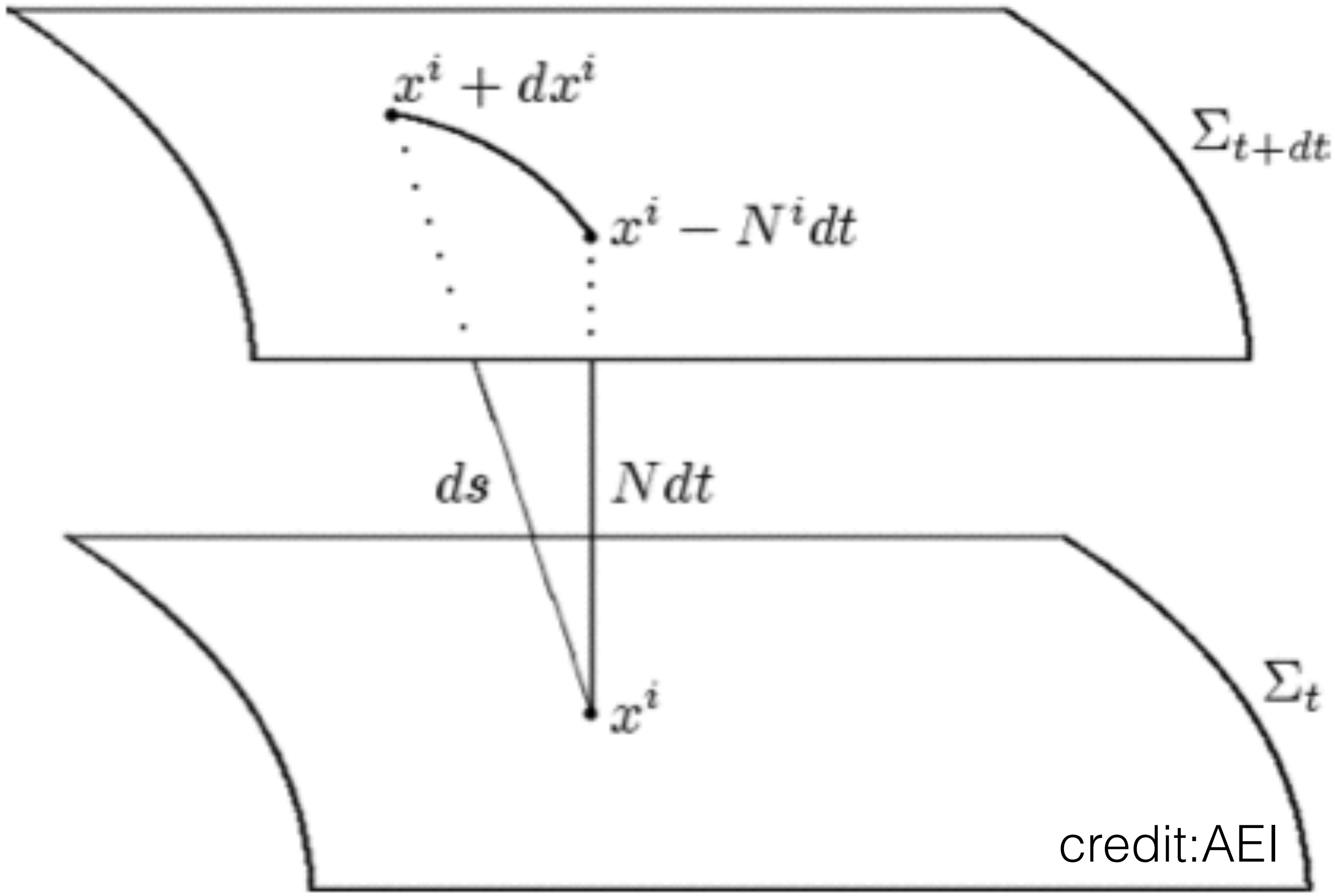
4–velocity $U^\mu = -g^{\mu\nu}\partial_\nu \chi$

- have to split space-time to define canonical variables

ADM split

after Richard Arnowitt, Stanley Deser and Charles W. Misner

- Foliate space-time into space-like hyper surfaces Σ_t



- metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}q^\mu \mathrm{d}q^\nu$

$$g_{00} = -N^2 + \gamma_{ij}N^iN^j \,, \qquad g_{0i} = \gamma_{ij}N^j \,, \qquad g_{ij} = \gamma_{ij}$$

N lapse, N^i shift

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ADM split II

- From the action we can calculate the canonical momenta

$$\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \dot{\gamma}_{ij}} = \frac{\sqrt{\gamma}}{2}\left(\gamma^{ij}K - K^{ij}\right) , \qquad \pi^{\chi} = \frac{\delta \mathcal{S}}{\delta \dot{\chi}} = \rho\sqrt{\gamma}\sqrt{1+\gamma^{ij}(\partial_i\chi)(\partial_j\chi)}$$

- extrinsic curvature $K_{ij} = \frac{1}{2N}\left[N_{i;j} + N_{j;i} - \dot{\gamma}_{ij}\right] , \qquad K = \gamma^{ij}K_{ij}$

covariant derivative w.r.t. γ_{ij}

- Action is now $\mathcal{S} = \int \mathrm{d}^4x \left[\phi^{\chi}\frac{\partial\chi}{\partial t} + \pi^{ij}\frac{\partial\gamma_{ij}}{\partial t} - \mathcal{H}'\right]$ with the "Hamiltonian density"

$$\mathcal{H}' \equiv N\mathcal{U} + N_i\mathcal{U}^i$$

$$\mathcal{U} = \pi^{\chi}\sqrt{1+\gamma^{ij}(\partial_i\chi)(\partial_j\chi)} + \frac{2}{\sqrt{\gamma}}\left[\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right] - \frac{\sqrt{\gamma}}{2}\left[{}^{(3)}R - 2\Lambda\right]$$

$$\mathcal{U}_i = -2\pi^k_{i;k} + \pi^{\chi}\partial_i\chi .$$

ADM split III

$$\mathcal{S} = \int \mathrm{d}^4x \left[\phi^{\chi} \frac{\partial \chi}{\partial t} + \pi^{ij} \frac{\partial \gamma_{ij}}{\partial t} - \mathcal{H}' \right]$$

- Varying this "new" action w.r.t π^{χ} and π^{ij} gives two evo eqn.

$$\left(\frac{\partial \chi}{\partial t} - N^i \partial_i \chi \right) = N \sqrt{1 + \gamma^{ij} (\partial_i \chi)(\partial_j \chi)} \,,$$

$$-K_{ij} = \frac{1}{\sqrt{\gamma}} \pi^{kl} \left(2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl} \right)$$

- $\boxed{\mathcal{H}' \equiv N\mathcal{U} + N_i \mathcal{U}^i}$ actually vanishes since the lapse and shift appear in the action as Lagrange multipliers. Their variation is

$$\mathcal{U} = 0 \,, \;\; \mathcal{U}_i = 0$$

- using these constraints, and specify the velocity potential χ to define the time hyper surfaces

$$\partial \chi / \partial t = 1, \boxed{N = 1, \; N^i = 0} \text{ and } \partial_i \chi = 0$$

"gauge choice" of initial data

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ADM split IV

- With these restrictions (valid for “dust”) the metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \gamma_{ij}(t, \boldsymbol{q})\, \mathrm{d}q^i \mathrm{d}q^j \,.$$

- Since the Hamiltonian vanishes, we can obtain directly the Hamilton-Jacobi theory of GR

$$\mathcal{S} = \mathcal{S}[\gamma_{ij}]$$

$$\frac{\partial \mathcal{S}}{\partial t} + H = 0$$

$$H = \int \mathrm{d}^3 q \left[\frac{2}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} \left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl} \right) - \frac{\sqrt{\gamma}}{2}(R - 2\Lambda) \right]$$

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PhD Core Course on General Relativity

LECTURE 3

Cornelius Rampf
ICG, University of Portsmouth

December 8-10-**12** 2014

ICG Portsmouth

Outline of the Course

- [x] Special Relativity and flat space-time
- [x] Manifolds
- [x] Curvature & Gravitation
- [x] Schwarzschild solution, black holes
- [x] ADM formalism
- [ ] "Weyl" formalism of GR
- [ ] Cosmological perturbations: standard, gradient expansion, …
- [ ] Eulerian/Lagrangian fluid descriptions in GR

LECTURE 3

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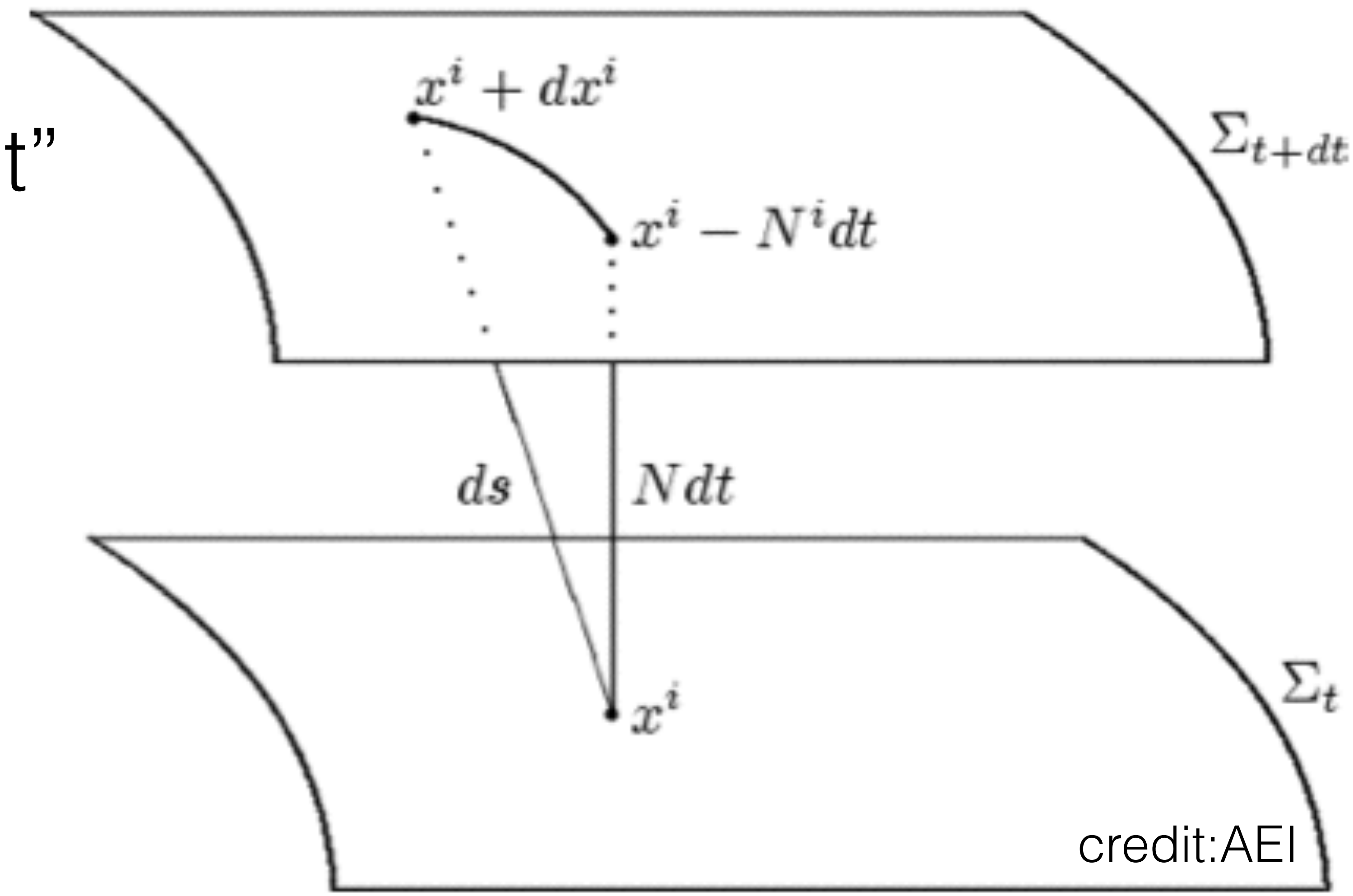
Last lecture we discussed…

☑ Field equations: $$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

☑ ADM split: Foliate space-time into space-like hyper surfaces Σ_t

"3+1 split"



☑ Hamiltonian, Hamilton-Jacobi theory of GR for $\Lambda\mathrm{CDM}$

Kundt & Trümper 1961, Ellis 1971

see also astro-ph/9403016

Alternative formulation of GR

- Standard way: Solve $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ for the metric, then Weyl tensor is fully determined!

$$C_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} + \frac{1}{2}\left(R_{\mu\lambda} g_{\nu\kappa} + R_{\nu\kappa} g_{\mu\lambda} - R_{\mu\kappa} g_{\nu\lambda} - R_{\nu\lambda} g_{\mu\kappa}\right) + \frac{1}{6} R \left(g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\kappa}\right)$$

- Alternative formulation: Weyl tensor is the fundamental geometrical quantity. Steps:

1. $\boldsymbol{\nabla}_\sigma R_{\mu\nu\kappa\lambda} + \boldsymbol{\nabla}_\mu R_{\nu\sigma\kappa\lambda} + \boldsymbol{\nabla}_\nu R_{\sigma\mu\kappa\lambda} = 0 \quad \Big| \cdot g^{\kappa\sigma}$
2. Use in 1.
3. Then convert the Riccis, $R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$

1+3 split

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}$$
$$g_{\mu\nu\gamma\delta} = g_{\mu\gamma}g_{\nu\delta} - g_{\mu\delta}g_{\nu\gamma}$$

- Yields field equations for the Weyl tensor:

$$\nabla^\kappa C_{\mu\nu\kappa\lambda} = 8\pi G \left(\nabla_{[\mu} T_{\nu]\lambda} + \frac{1}{3} g_{\lambda[\mu} \nabla_{\nu]} T^\kappa{}_\kappa \right)$$

- Note that Weyl tensor is fully described by

electric part, magnetic part

$$E_{\mu\nu}(u) \equiv u^\kappa u^\lambda C_{\mu\kappa\nu\lambda} \,, \quad H_{\mu\nu}(u) \equiv \frac{1}{2} u^\kappa u^\lambda \epsilon_{\alpha\beta\kappa(\mu} C^{\alpha\beta}{}_{\nu)\lambda}$$

$$C_{\mu\nu\kappa\lambda} = (g_{\mu\nu\alpha\beta}\, g_{\kappa\lambda\gamma\delta} - \epsilon_{\mu\nu\alpha\beta}\, \epsilon_{\kappa\lambda\gamma\delta})\, u^\alpha u^\gamma E^{\beta\delta}(u) + (\epsilon_{\mu\nu\alpha\beta}\, g_{\kappa\lambda\gamma\delta} + g_{\mu\nu\alpha\beta}\, \epsilon_{\kappa\lambda\gamma\delta})\, u^\alpha u^\gamma H^{\beta\delta}(u)$$

→ Fully covariant Maxwell-like field equations

Lagrangian trajectory of a fluid element

$$x^\mu = x^\mu(q^i, \tau) \,, \quad u^\mu = \frac{\partial x^\mu(q^i, \tau)}{\partial \tau}$$