1+3 split

Kundt & Trümper 1961, Ehlers 1961, Ellis 1971

see also astro-ph/9403016

- Could write down the covariant field equations for a CDM fluid

- This split is however most conveniently expressed in its fluid frame (locally inertial frame, $u^\mu = (1,0,0,0)$)

$$(\text{div-}E): \quad \nabla_j E^j{}_i - \epsilon_{ijk}\sigma^{jl}H^k{}_l - 3H_{ij}\omega^j = \frac{8\pi}{3}G\nabla_i\rho \ ,$$

$$(\dot{H}): \quad \frac{dH_{ij}}{dt} + \nabla_k\,\epsilon^{kl}{}_{(i}E_{j)l} + \Theta H_{ij} + \delta_{ij}\,\sigma^{kl}H_{kl} - 3\sigma^k{}_{(i}H_{j)k} - \omega^k{}_{(i}H_{j)k} = 0 \ ,$$

$$(\text{div-}H): \quad \nabla_j H^j{}_i + \epsilon_{ijk}\sigma^{jl}E^k{}_l + 3E_{ij}\omega^j = -8\pi G\rho\,\omega_j \ ,$$

$$(\dot{E}): \quad \frac{dE_{ij}}{dt} - \nabla_k\,\epsilon^{kl}{}_{(i}H_{j)l} + \Theta E_{ij} + \delta_{ij}\,\sigma^{kl}E_{kl} - 3\sigma^k{}_{(i}E_{j)k} - \omega^k{}_{(i}E_{j)k} = -4\pi G\rho\,\sigma_{ij}$$

credit:Hamilton&Bertschinger

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Gravitational waves

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- GW = **dynamical** tensor perturbation
- Gravitational information propagates with speed of light
  → gravitational waves naturally arise in a Lorentz invariant tensor theory of gravity!
- Non-existent in Newtonian gravity
- Indirect evidence of GW's from measurements of the Hulse-Taylor binary system

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Gauges, solution techniques

Coordinates in GR

- GR is formulated in a coordinate invariant way

$$x^\mu \rightarrow x^\mu + \zeta^\mu$$

$x^\mu \rightarrow x^\mu + \zeta^\mu$	- 4 dof
metric	+10 dof
	6 dof

- to derive any quantity, we should fix the coordinate system
- the resulting coordinate conditions are called gauge
- there are infinitely possible gauge choices; one should pick a given gauge to maximise the physical interpretation

What to do with the **6 dof**?

- line element with **not yet specified** gauge conditions

$$\mathrm{d}s^2 = -(1+2A)\mathrm{d}t^2 + 2a(t)w_i\mathrm{d}t\mathrm{d}x^i + a^2(t)T_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

scalar

scalar and vector

use spatial comoving coordinates (for later convenience)

pure tensor

$$T_{ij} = \frac{\delta_{ij}}{3}\hat{Q} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\boldsymbol{\nabla}^2\right)\hat{T}^{\parallel} + 2\hat{T}^{\perp}_{(i,j)} + \hat{T}^{\mathrm{T}}_{ij}$$

- 4 scalars, 2 vectors (= 4 dof), and one tensor (2 dof) = **10 dofs**

$w_i = \boldsymbol{\nabla}_{\boldsymbol{x}} S + \boldsymbol{\nabla}_{\boldsymbol{x}} \times \boldsymbol{T}$

here comma and ∇ denote partial derivatives

Gauges in GR

- GR is constructed in a gauge-invariant way
- Field equations deliver 6 dynamical equations
- We thus should fix a coordinate system
- Gauge-invariant formalism (Bardeen 1982, see next slide)
- HOWEVER, a measurement is generally gauge-dependent because the observer has to choose a frame (e.g., Lagrangian vs Eulerian frame of reference)

Bardeen 1980, Mukhanov, Feldman & Brandenberger 1992

Gauge-invariant formalism

- Let's focus on linear scalar perturbations only

$$ds^2 = a^2(\eta)\{(1+2\phi)\,d\eta^2 - 2B_{|i}\,dx^i\,d\eta - [(1-2\psi)\gamma_{ij} + 2E_{|ij}]\,dx^i\,dx^j\}$$

here partial derivative

set $\gamma_{ij} := \delta_{ij}$

- Study infinitesimal coordinate trafo $x^\alpha \rightarrow \tilde{x}^\alpha = x^\alpha + \xi^\alpha$

$$\eta \rightarrow \tilde{\eta} = \eta + \xi^0(\eta, \mathbf{x}), \qquad x^i \rightarrow \tilde{x}^i = x^i + \gamma^{ij}\xi_{|j}(\eta, \mathbf{x})$$

$$\longrightarrow \tilde{\phi} = \phi - (a'/a)\xi^0 - \xi^{0\prime}, \quad \tilde{\psi} = \psi + (a'/a)\xi^0, \quad \tilde{B} = B + \xi^0 - \xi', \quad \tilde{E} = E - \xi$$

- One of the **infinite** possible gauge-invariant combinations are

$$\Phi = \phi + (1/a)[(B - E')a]', \qquad \Psi = \psi - (a'/a)(B - E')$$

Some gauge choices:

- Synchronous/comoving gauge

$$ds^2 = -dt^2 + a^2 T_{ij} dq^i dq^j \quad \text{with} \quad T_{ij} = \frac{\delta_{ij}}{3}\hat{Q} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\boldsymbol{\nabla}^2\right)\hat{T}^{\parallel} + 2\hat{T}^{\perp}_{(i,j)} + \hat{T}^{\mathrm{T}}_{ij}$$

is the Lagrangian frame (see later)

- Poisson gauge (Newtonian gauge)

$$ds^2 = -(1+2A)dt^2 + 2a(t)w_i dt dx^i + a^2(t)\left([1-2B]\delta_{ij} + T^{\mathrm{T}}_{ij}\right)dx^i dx^j$$

here only a vector (w_i) — pure tensor (T^{T}_{ij})

Has a “nice” Newtonian limit in the scalar sector

Scalar, vector, tensor decomposition

- decouple only in linear theory
- non-linear decoupled (see e.g., perturbation theory)
- e.g., scalars induce vector and tensor perturbations
- cosmology: negligible vector and tensor perturbations at recombination

Standard perturbation theory

1. Assume initial conditions
   (structure formation: Bardeen's scalar potential $\Phi \sim 10^{-5}$,
   no vectors/tensors at initial time)

2. choose a gauge

3. Ansatz for all variables: $A = \sum_{n=1} A^{(n)} \equiv \sum_{n=1} f(\Phi^n)$

4. Solve Einstein equations to first order

5. Iterate, $A^{(n)} = \mathcal{F}(A^{(m<n)}, B^{(m<n)}, \cdots)$, ...

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Solve Einstein equations in a series
with increasing number of spatial gradients:

Gradient expansion technique

1. Choose synchronous/comoving coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \gamma_{ij}\,\mathrm{d}q^i\mathrm{d}q^j$$

2. Ansatz: $\frac{1}{a}\partial_k\gamma_{ij} \ll \partial_t\gamma_{ij}$, i.e., assume that spatial gradients do not dominate the time evolution of the spatial metric

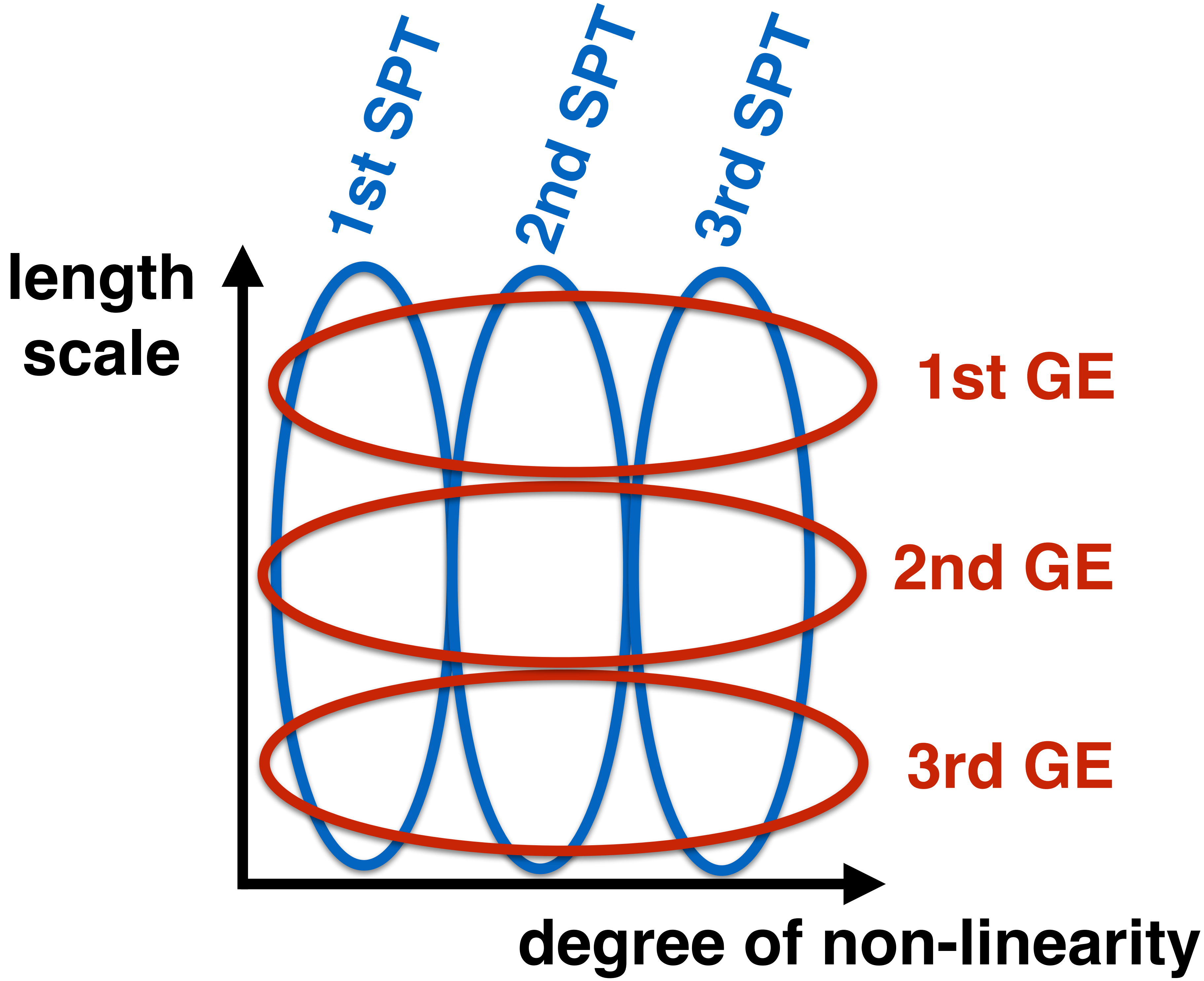
$$\gamma_{ij} = \sum_n \gamma_{ij}^{(n)} \equiv \sum_n f_{ij}(\nabla^n)$$

3. Initial conditions: $\gamma_{ij}^{(0)} = \delta_{ij}\left(1 + \frac{10}{3}\Phi\right)$

4. Plug this into the field equations and let
$R_{ij} \doteq \partial_i\partial_j\gamma + (\partial_i\gamma)(\partial_j\gamma)$ etc. do its business!

5. Iterate

gradient expansion (**GE**)
vs.
standard perturbation theory (**SPT**)



Post-Newtonian expansion

- Expansion parameter is $1/c^2$ $\qquad (c \to \infty)$

  (no assumption on the vanishing curvature)

- E.g., in the Poisson gauge (a Eulerian frame, see later)

$$ds^2 = a^2(\eta)\left[-\left(1+\frac{A^{(1)}}{c^2}+\frac{A^{(2)}}{c^4}\right)c^2 d\eta^2 + \frac{w_i^{(2)}}{c^3} c\, d\eta\, dx^i + \left(1+\frac{B^{(2)}}{c^2}\right)\delta_{ij}dx^i dx^j\right]$$

transverse vector

(no tensor perturbations up to this order)

- In the synchronous/comoving gauge

$$ds^2 = a^2(\tau)\left[-c^2 d\tau^2 + \left(\gamma_{ij}^{(1)} + \frac{\gamma_{ij}^{(2)}}{c^2}\right)dq^i dq^j\right]$$

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Newtonian and Post-Newtonian Limits of Relativistic Cosmology

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The Newtonian limit of general relativity is by no means as straightforward as is commonly assumed. In particular, the correct limit of the Bianchi identities must be taken to the second (non-linear) order. Furthermore Newtonian cosmology does not have a well-posed initial value formulation, while relativistic cosmology does. We show in this paper that the c^{-4} approximation of general relativity, although non-linear, provides a non-standard version of Newtonian theory which is in fact completely equivalent to the Heckmann–Schücking version of Newtonian cosmology. The next approximation (order c^{-6}), when the limit is taken in a particular way, gives rise to a closed and self-consistent post-Newtonian cosmological theory which has a well-posed initial value problem. This seems to be a suitable, if somewhat complicated, theory for cosmological and astrophysical problems.

KEY WORDS : Well-posed Cauchy problem

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