Eulerian & Lagrangian flows in **Newtonian cosmology**

(in a nutshell)

The Eulerian way

density contrast $\delta = (\rho - \bar{\rho})/\bar{\rho}$

peculiar velocity $a\boldsymbol{u} = \boldsymbol{U} - \dot{a}\boldsymbol{x}$

Background evolution à la Friedmann $$\left(\frac{\dot{a}}{a}\right)^2 = a^{-3} + \Lambda\,,$$

Newtonian fluid equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}})\,\boldsymbol{u} = -2\frac{\dot{a}}{a}\boldsymbol{u} - \frac{1}{a^2}\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi_g\,,$$

$$\partial_t\delta + \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot[(1+\delta)\boldsymbol{u}] = 0\,,$$

$$\boldsymbol{\nabla}^2_{\boldsymbol{x}}\varphi_g = \frac{3}{2a}\delta\,,$$

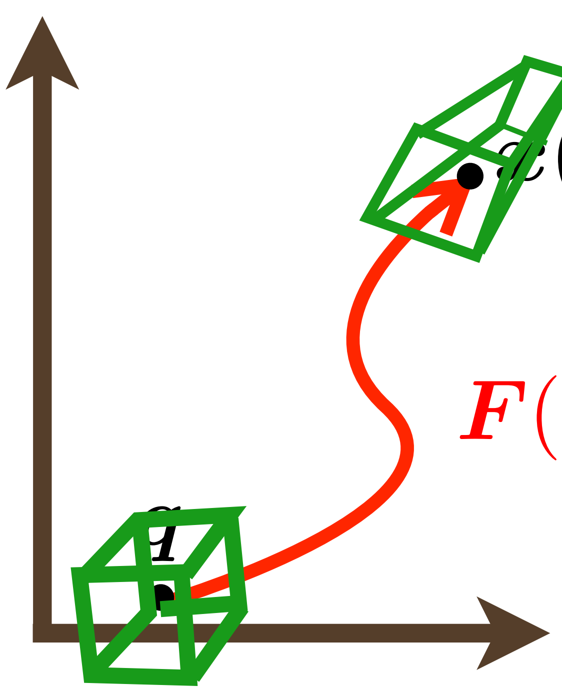
Eulerian Perturbation Theory: $$\delta = \sum_n \delta^{(n)}\,, \quad \boldsymbol{u} = \sum_n \boldsymbol{u}^{(n)}\,.$$

review: astro-ph/0112551

The Lagrangian way

Coordinate trafo: $\boldsymbol{x}(t,\boldsymbol{q}) = \boldsymbol{q} + \boldsymbol{F}(t,\boldsymbol{q})$

Lagrangian coordinate



$\boldsymbol{x}(t,\boldsymbol{q})$ Eulerian field coordinate

Observer's / Eulerian frame

$\boldsymbol{F}(t,\boldsymbol{q})$ displacement field
(contains all the dynamical info!)

Lagrangian density $\delta(t,\boldsymbol{q}) = \left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}\right|^{-1} - 1$

$$\boldsymbol{F}(t,\boldsymbol{q}) = \sum_n \boldsymbol{F}^{(n)}(t,\boldsymbol{q})$$

Lagrangian velocity $\boldsymbol{u} = D_t \boldsymbol{F}$

Lagrangian & Eulerian approaches in General Relativity

Or: How to generalise the above to GR?

The **Lagrangian** frame in GR

[Ehlers'61, Matarrese&Terranova'96, CR&Rigopoulos'13]

◆First task: **Define** the corresponding coordinate system to be synchronous and comoving

$$ds^2 = -dt^2 + \gamma_{ab}\, dq^a dq^b \,, \qquad q^a \textbf{ label the fluid elements}$$

◆The **4-trajectory** of the fluid element is then $x^\mu = x^\mu\,(t, q^a)$ (x^μ are the space-time coordinates associated with a not yet specified coordinate system)

◆This is nothing but a coordinate/gauge transformation!

$$x^\mu(t, \boldsymbol{q}) = q^\mu + F^\mu(t, \boldsymbol{q}) \qquad x^\mu = \begin{pmatrix} \tau \\ \boldsymbol{x} \end{pmatrix} ;\; q^\mu = \begin{pmatrix} t \\ \boldsymbol{q} \end{pmatrix} ;\; F^\mu = \begin{pmatrix} L \\ \boldsymbol{F} \end{pmatrix} ;$$

compare with 1+3 split!

Eulerian frames in GR

- Define **any** Eulerian frame: The spatial part of F^{μ} is the Lagrangian 3-displacement field, **if** it carries the longitudinal & transverse part of the Lagrangian metric

Decomposition: $$T_{ij} = \frac{\delta_{ij}}{3}\hat{Q} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right)\hat{T}^{\parallel} + 2\hat{T}^{\perp}_{(i,j)} + \hat{T}^{\mathrm{T}}_{ij},$$

$$\Rightarrow \quad \delta^{\mathbf{E}}(t,\boldsymbol{x}) = \delta^{\mathbf{E}}(t,\boldsymbol{q}+\boldsymbol{F}) = \delta^{\mathbf{L}}(t,\boldsymbol{q}) + \boldsymbol{F}\cdot\nabla\delta^{\mathbf{L}} + \cdots$$

- A **specific** Eulerian frame is then obtained by fixing the temporal gauge condition, i.e., fix F^0

$$x^{\mu}(t,\boldsymbol{q}) = q^{\mu} + F^{\mu}(t,\boldsymbol{q})$$

Example 1: **Poisson gauge**

1. Start with a **relativistic solution** in the Lagrangian frame, obtained from e.g., the gradient expansion technique or conventional PT (up to second order)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \gamma_{ab}\,\mathrm{d}q^a\mathrm{d}q^b$$

2. Perform gauge trafo $x^\mu(t,\boldsymbol{q}) = q^\mu + F^\mu(t,\boldsymbol{q})$ to the Poisson gauge

$$\mathrm{d}s^2 = -(1+2A)\mathrm{d}\tau^2 + 2aw_i\mathrm{d}\tau\mathrm{d}x^i + a^2([1-2B]\delta_{ij} + \chi_{ij})\mathrm{d}x^i\mathrm{d}x^j$$

A, B scalars, w_i transverse vector, χ_{ij} pure tensor

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g_{\mu\nu}$$

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Example 1: **Poisson gauge**

$$x^\mu(t, \boldsymbol{q}) = q^\mu + F^\mu(t, \boldsymbol{q})$$

… And the (2nd order) result is:

- F^a is the Newtonian displacement field + GR corrections
- F^0 is the Newtonian velocity potential + GR corrections
- We obtain the **Eulerian density** + GR corrections
- GR Vector perturbation in F^a induces frame dragging
- GR tensor perturbations are only gravitational waves

… OK, but what does the above mean & imply?

- GR corrections are generally small but become important on scales close to the causal horizon
- Most GR corrections result because scalars, vectors and tensors do no decouple beyond leading order

What we have learnt… (?)

- In GR one can obtain a Newtonian limit in
  - In a unique Lagrangian frame
  - In an infinite class of Eulerian frames

→ GR differs from Newton generally leading leading order

- It is possible to trim GR to deliver exactly the Newtonian approximation, but not without doing serious damage to GR (see Szekeres 1999)