# Theoretical Cosmology

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ICG Lectures, 2015

# Outline

- Lecture 1: Overview and the Expansion
- Lecture 2: The Early Universe
- Lecture 3: Problems with the Standard Model of Cosmology

# The Cosmological Principle

- Copernican principle: we do not occupy a special place in the Universe
  - Philosophical principle
- Cosmological principle: Universe is homogeneous and isotropic *on very large scales*
  - Homogeneous = same everywhere
  - Isotropic = same in all directions
  - Can be tested by large-scale surveys
  - Applies on scales > 10s of Mpc (or 100s?)
- We observe isotropy, and assuming isotropy everywhere via the Copernican principle implies homogeneity

# Key Features of the Big Bang Model

1. Hubble expansion
   - Lecture 1

2. Existence of microwave background radiation
   - Lecture 2

3. Prediction of primordial nucleosynthesis
   - Lecture 2

Velocity

500 km/s

$$v = H_0 d$$

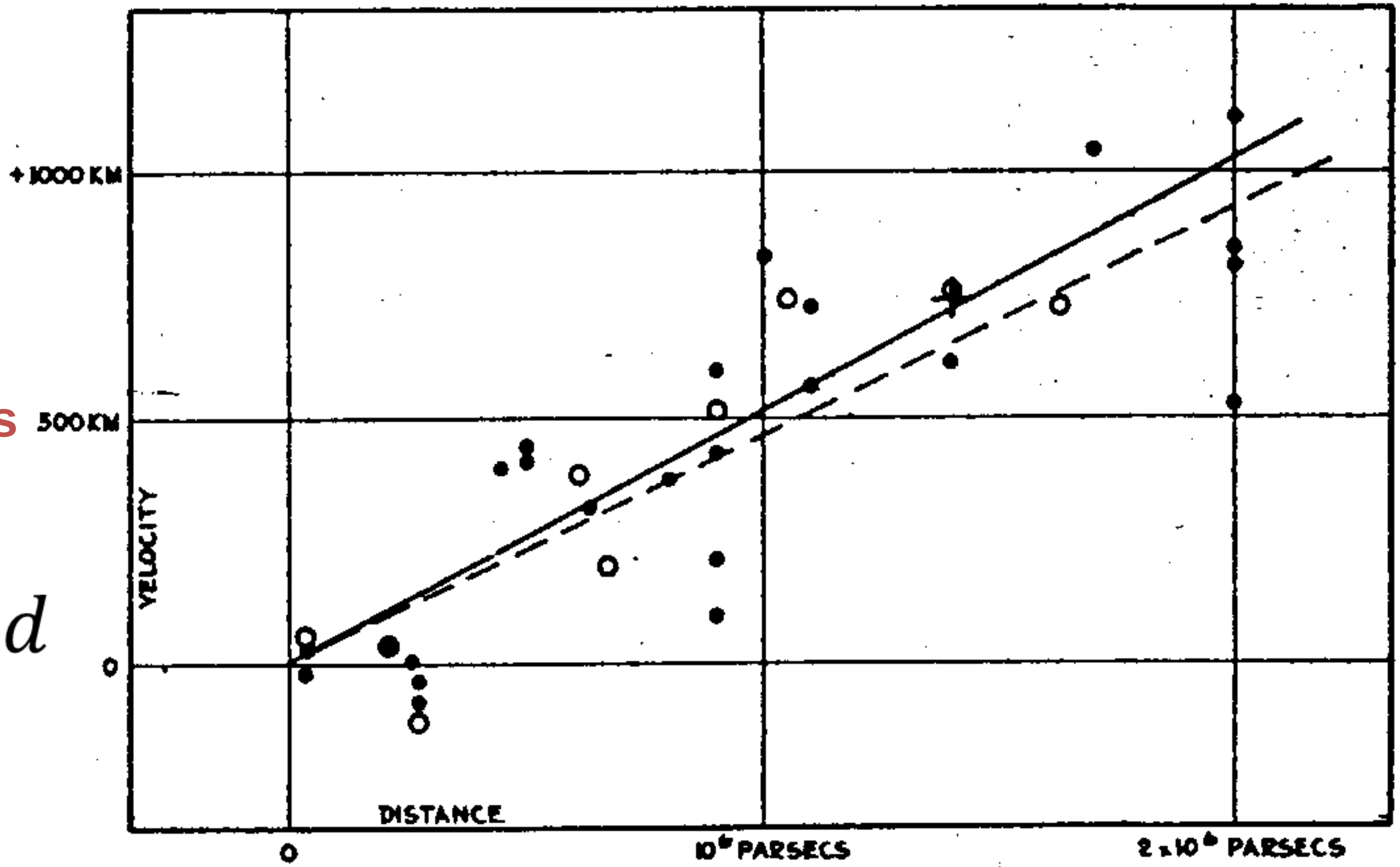
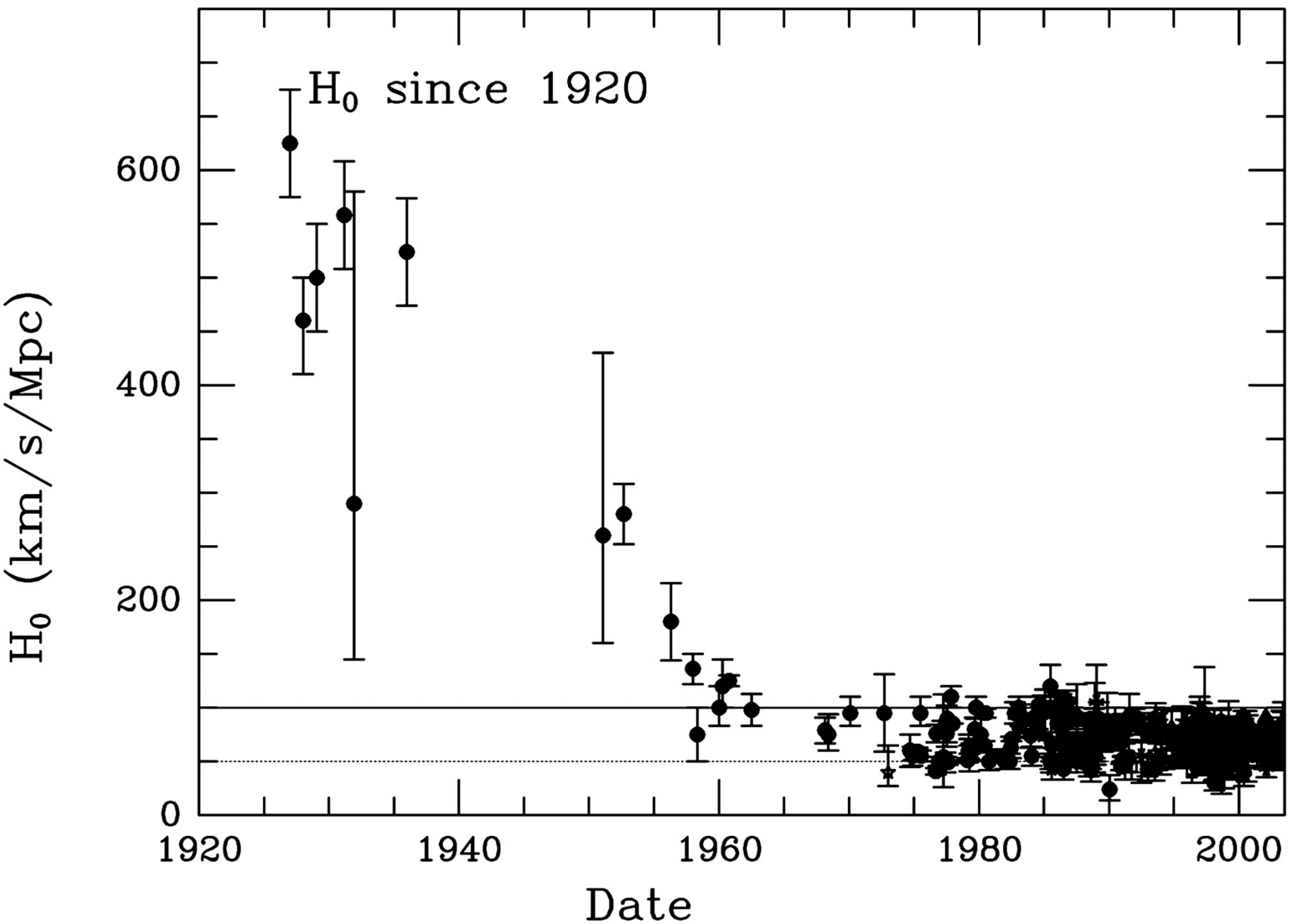


FIGURE 1

1 Mpc

Distance

$$H_0 = 500 \text{ km/s/Mpc ??}$$



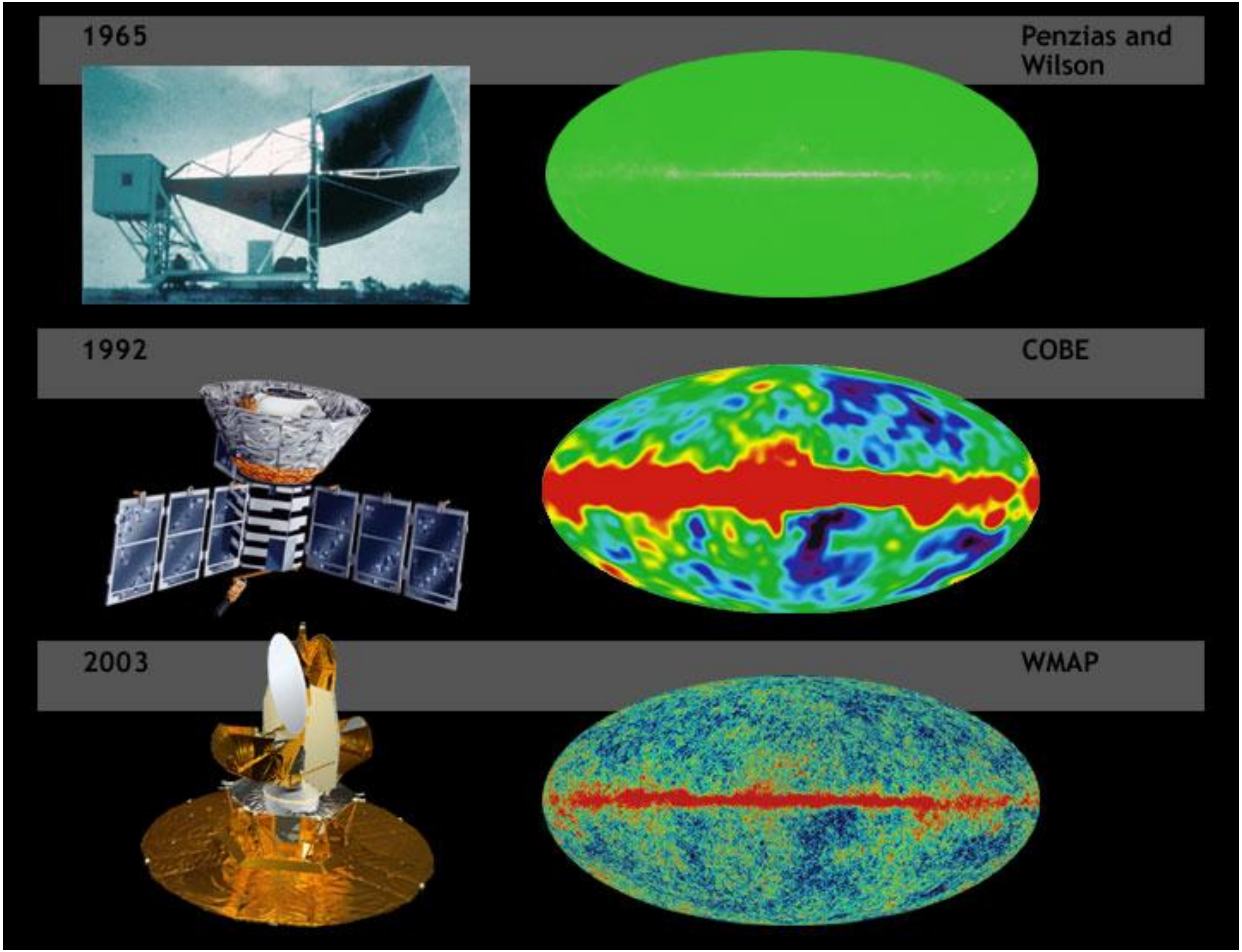
$H_0$ since 1920
600
400
200
0
1920
1940
1960
1980
2000
Date
$H_0$ (km/s/Mpc)

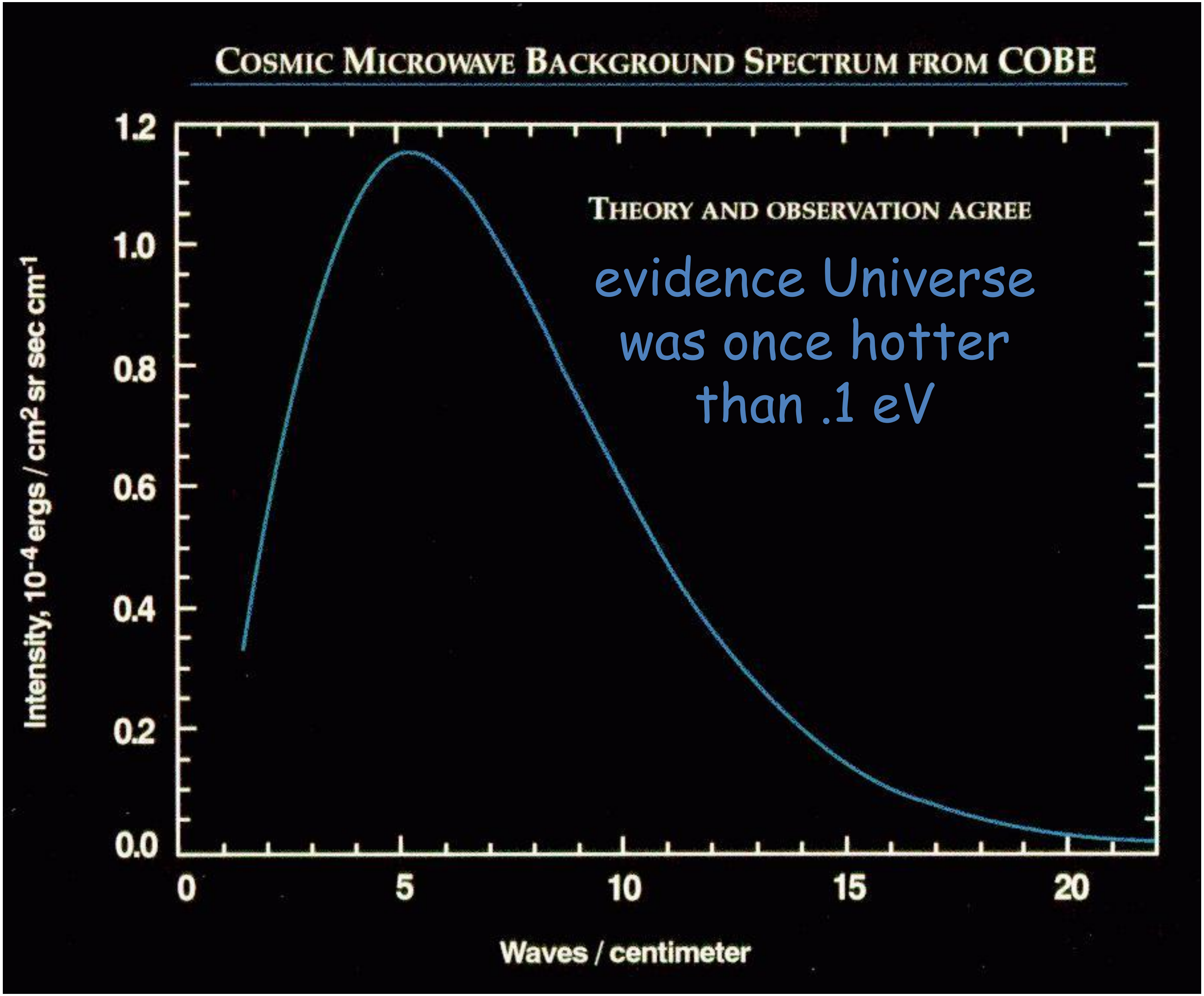
# Implications of Expansion

- Big Bang model: the Universe is expanding now, so it was very small in the past, and had a “beginning”
- Steady State model: matter is continually created as space expands, so the Universe is homogeneous in time as well as space

*Either interpretation valid until the discovery of the microwave background radiation, which favored BB*

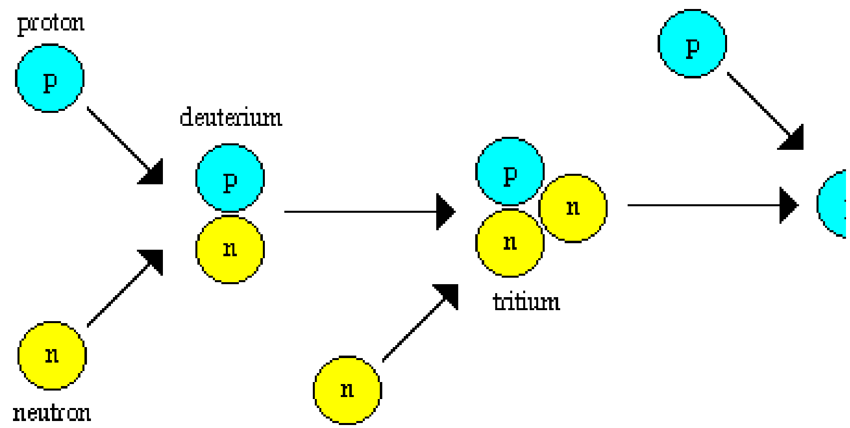
# Cosmic Microwave Background (CMB)



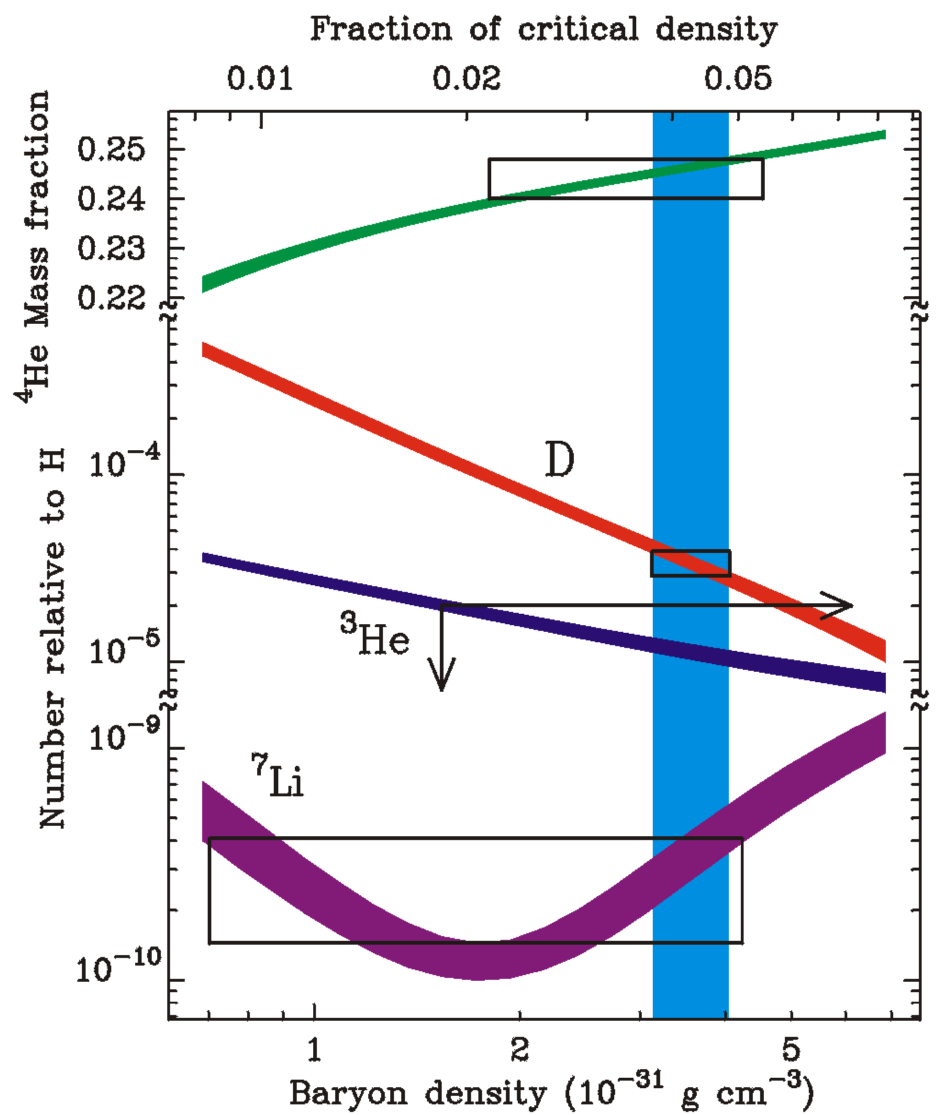



evidence Universe
was once hotter
than .1 eV

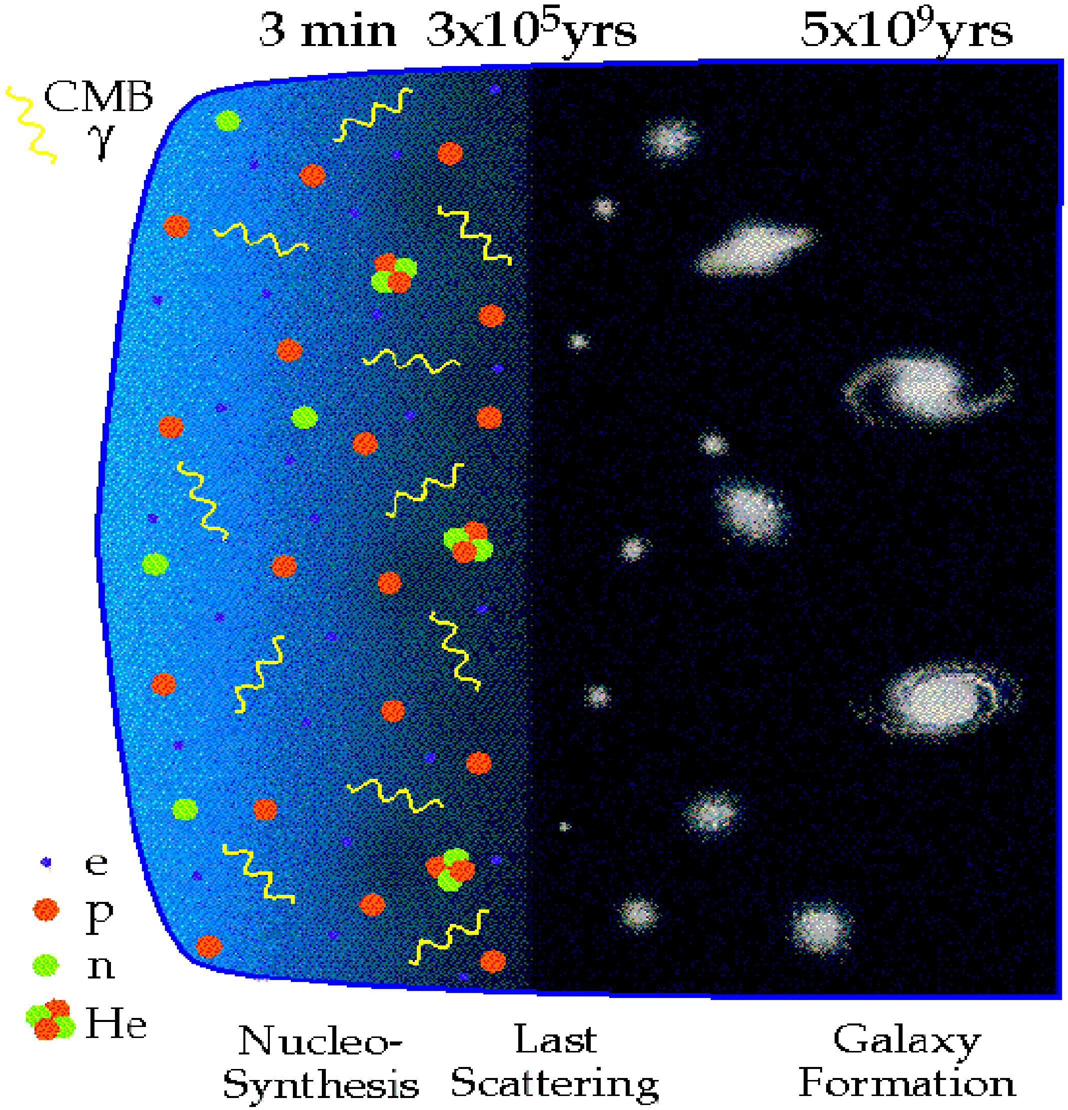
# Primordial Nucleosynthesis




A prediction of the hot big bang is the production of light elements through nuclear reactions in the initial plasma



# (Very) Brief History



# Big Bang Cosmology

- Cosmological principle: the Universe is homogeneous and isotropic
  - (on very large scales)

- The Big Bang: the Universe was very small, hot, and dense in the past

  - Hubble expansion

  - The Cosmic Microwave Background

  - Primordial Nucleosynthesis (or BBN)

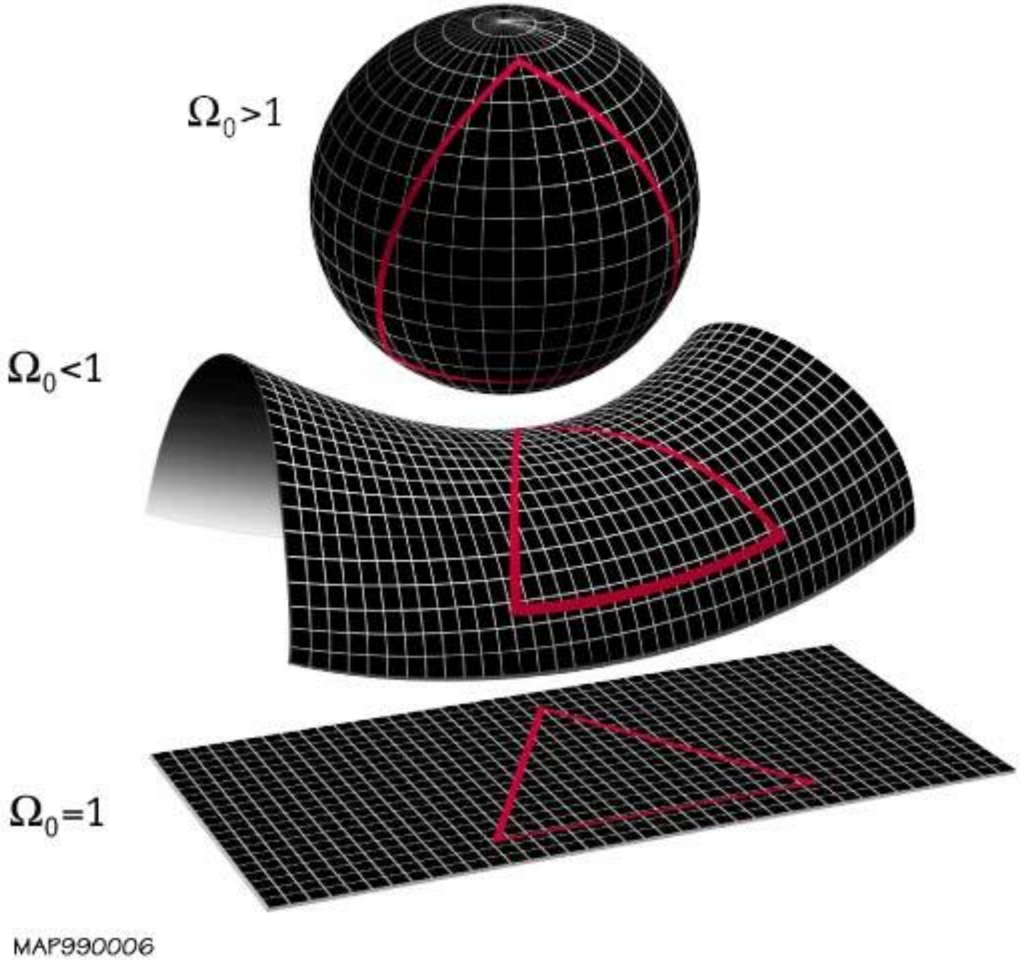
# The Expanding Universe

Geometry
Dynamics
Kinematics
Thermodynamics

# Geometry

- A *metric* allows us to calculate the spacetime separation between 2 (4-D) points in a given coordinate system

- 3-dimensional space has three possible geometries:
  - Spherical
  - Hyperbolic
  - Euclidean




*Possible geometries of 2-D surfaces*

- In flat space, can expand Minkowski metric of special relativity to allow for expansion:

$$ds^2 = -c^2dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

- Allowing for curvature (and in spherical coordinates):

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

where

$$\begin{cases} K = 1 & \text{Spherical} & \text{“closed”} & \Omega > 1 \\ K = 0 & \text{Euclidean} & \text{“flat”} & \Omega = 1 \\ K = -1 & \text{Hyperbolic} & \text{“open”} & \Omega < 1 \end{cases}$$

This FLRW metric (Friedmann, Lamaitre, Robertson, and Walker) is the most general metric for a homogeneous and isotropic universe.

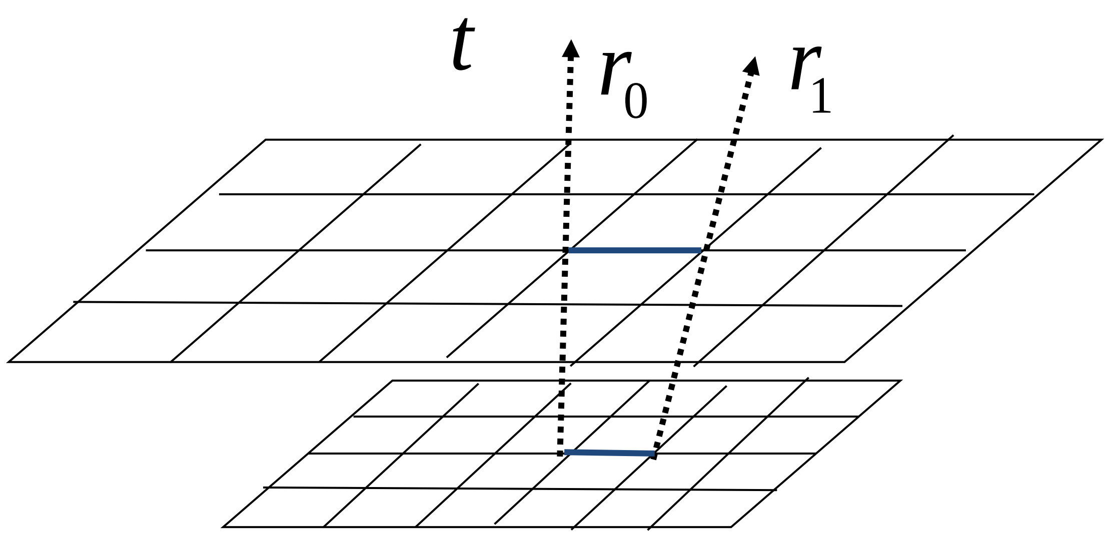
- Can also write in terms of the conformal time:

$$d\tau \equiv \frac{dt}{a(t)},$$

$$ds^2 = a^2(\tau)\left[-c^2 d\tau^2 + \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

- Regions separated by distance $> c\tau$ are not causally connected

- Note that $r$ is a co-moving coordinate:



# Dynamics

- In GR, the metric of space-time is related to the matter and energy in the Universe.

$$ds^2 = g_{\mu\nu} x^\mu x^\nu$$

- The Einstein equation (now $c = 1$) is a second order differential equation for $a(t)$:

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R} = \boxed{8\pi G T_{\mu\nu} -}\boxed{\Lambda g_{\mu\nu}}$$

geometry stuff vacuum energy

- Perfect fluids have $T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, p, p)$

- Using FLRW metric, there are two independent Einstein equations (time-time and space-space components), which give the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

- Can derive a third (not independent) describing energy-momentum conservation, giving the fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

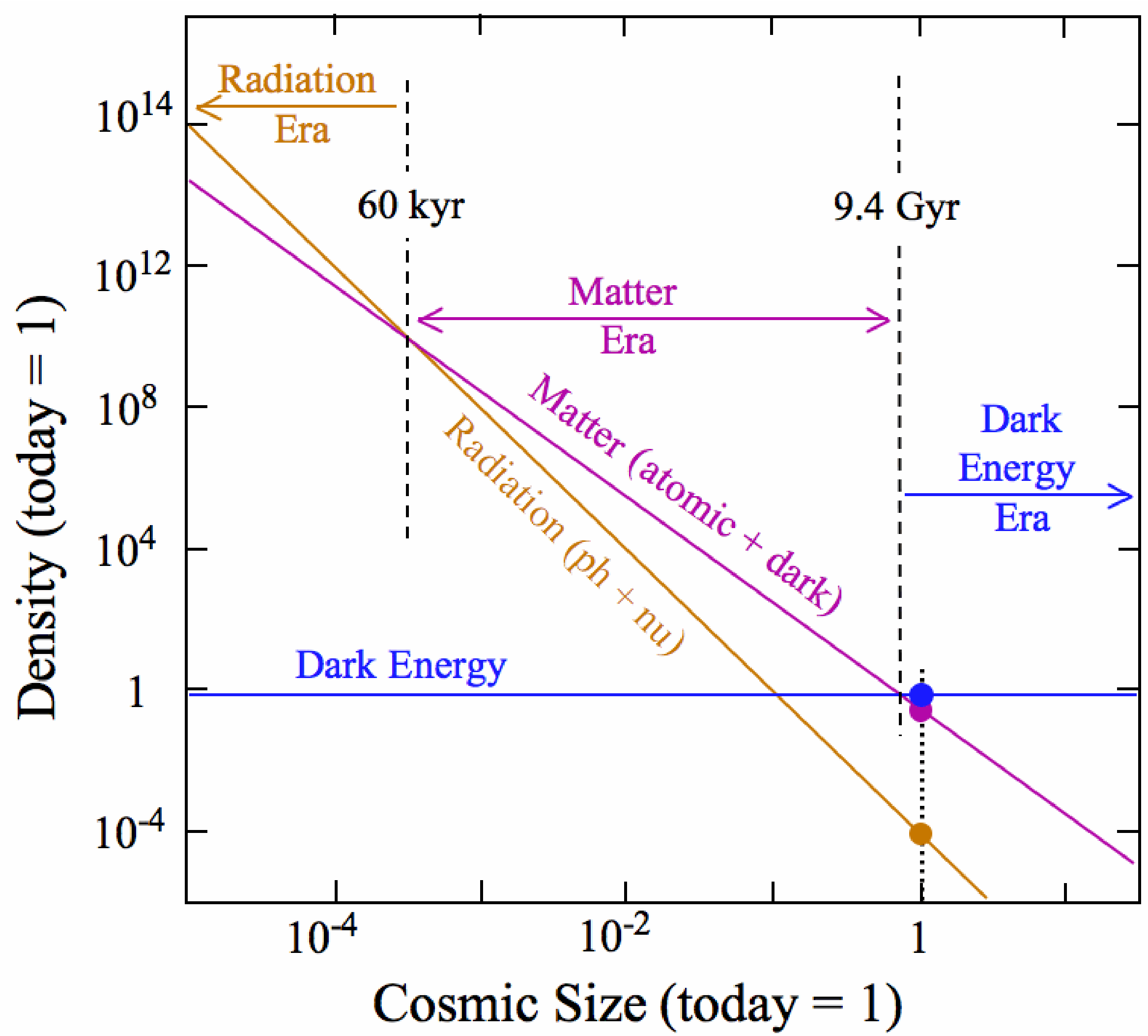
- Now we have 2 (independent) equations and 3 unknowns: $a(t), \rho(t), p(t)$

- Need "equation of state" relating ρ and $\rho$
  - Radiation: $p = \rho/3$
  - Matter: $p = 0$
  - Vacuum energy: $p = -\rho$
  - In general: $p = w\rho$

- Example: Einstein-de Sitter ($K = 0, \Lambda = 0$):

$$\rho \propto a^{-3(1+w)}, a \propto t^{\frac{2}{3(1+w)}}$$

  - Radiation: $w = 1/3 \Rightarrow \rho \propto a^{-4}$ and $a \propto t^{1/2}$
  - Matter: $w = 0 \Rightarrow \rho \propto a^{-3}$ and $a \propto t^{2/3}$



Radiation
Era
60 kyr
9.4 Gyr
Matter
Era
Matter (atomic + dark)
Radiation (ph + nu)
Dark
Energy
Era
Dark Energy
$10^{14}$
$10^{12}$
$10^{8}$
$10^{4}$
1
$10^{-4}$
Density (today = 1)
$10^{-4}$
$10^{-2}$
1
Cosmic Size (today = 1)

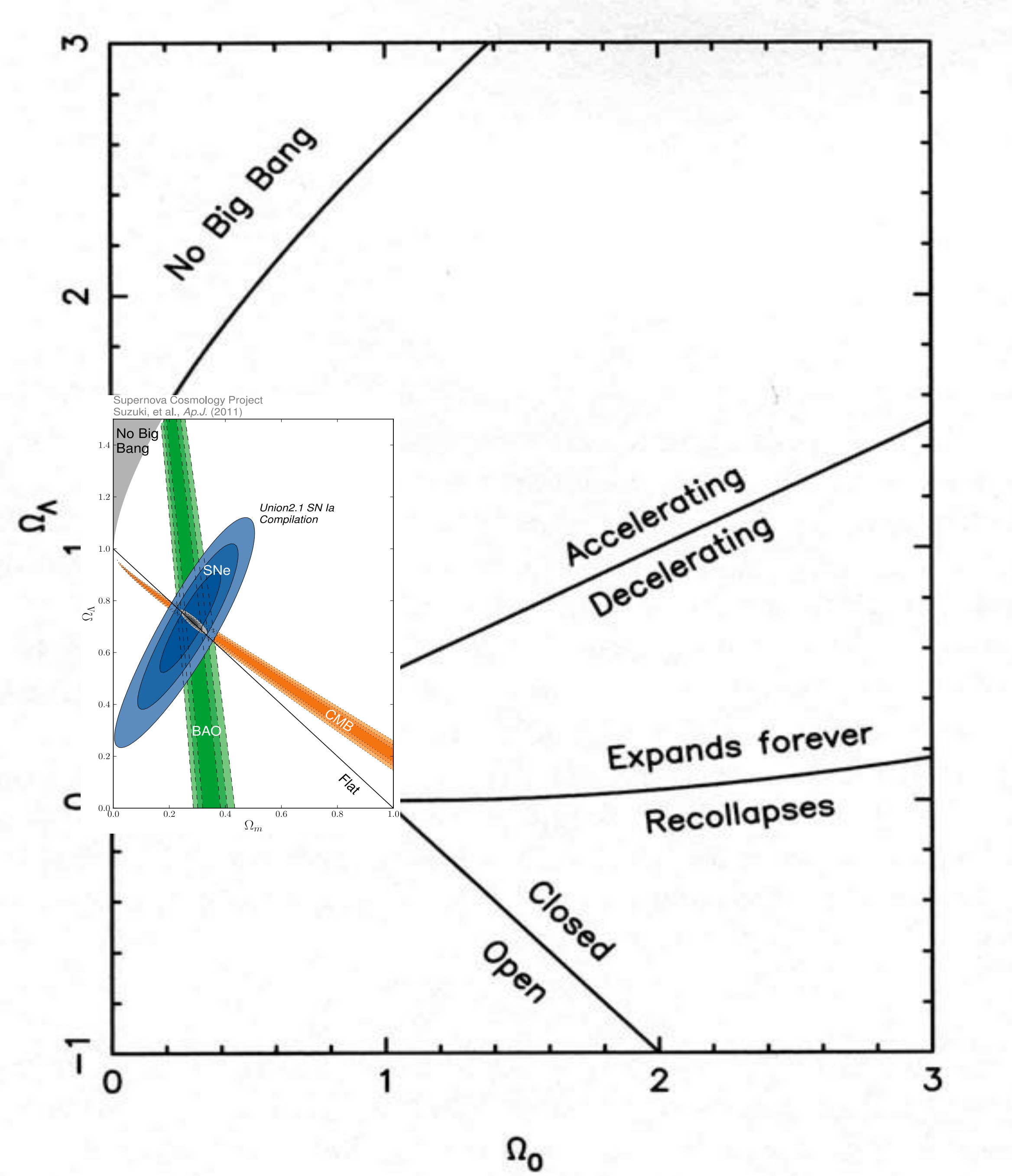
- Note that $\vec{v} = \frac{|\dot{\vec{r}}|}{|\vec{r}|}\vec{r} = \frac{\dot{a}}{a}\vec{r}, v = Hr \rightarrow H = \frac{\dot{a}}{a}$

- Neglecting $\Lambda$, a critical density would make $K = 0$:
$$\rho_c \equiv \frac{3H^2}{8\pi G}$$

- Defining density parameter $\Omega = \rho/\rho_c$ you can write the Friedman equation as:
$$\Omega = 1 + \frac{K}{a^2 H^2}$$

Thus $\Omega > 1, \Omega = 1$, and $\Omega < 1$ are open, flat, and closed.

- Adding back $\Lambda$ with $\Omega_\Lambda = \Lambda/3H^2$, then $\Omega_m + \Omega_\Lambda = 1$ in a flat universe.




The relative amounts of dark matter and dark energy determine the global curvature and expansion history of the Universe.

# Kinematics

- Light propagates along a null line where $ds^2 = 0$. Consider light (or peak of wave) emitted at $t_e$ and $t_e + \Delta t_e$:

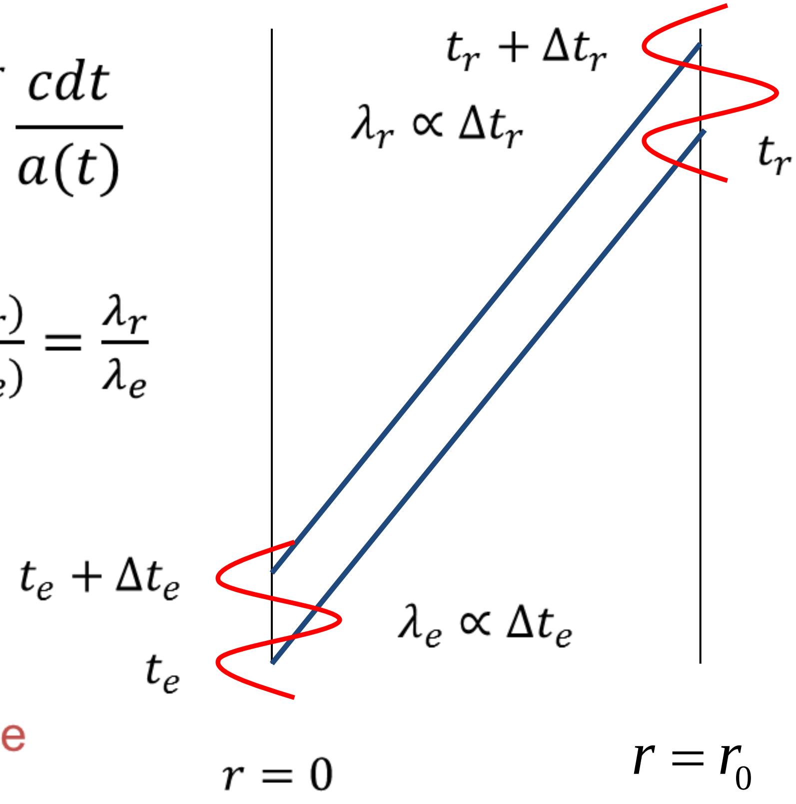
$$\int_{t_e}^{t_r} \frac{cdt}{a(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1 - Kr^2}} = \int_{t_e+\Delta t_e}^{t_r+\Delta t_r} \frac{cdt}{a(t)}$$

- Then for $\Delta t \ll 1$: $\frac{\Delta t_r}{a(t_r)} = \frac{\Delta t_e}{a(t_e)} \Rightarrow \frac{a(t_r)}{a(t_e)} = \frac{\lambda_r}{\lambda_e}$

- Defining for us today: $a(t_r) = 1$

$$\Rightarrow \frac{1}{a} = \frac{\lambda_r}{\lambda_e} = 1 + z$$

So the redshift $z$ is the fractional change in wavelength of light due to expansion



- The line-of-sight co-moving distance to light emitted at time $t_e$ is
$$d_c = \int_{t_e}^{t_0} \frac{cdt}{a} = \int_{a_e}^{1} \frac{cda}{Ha^2} = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}$$

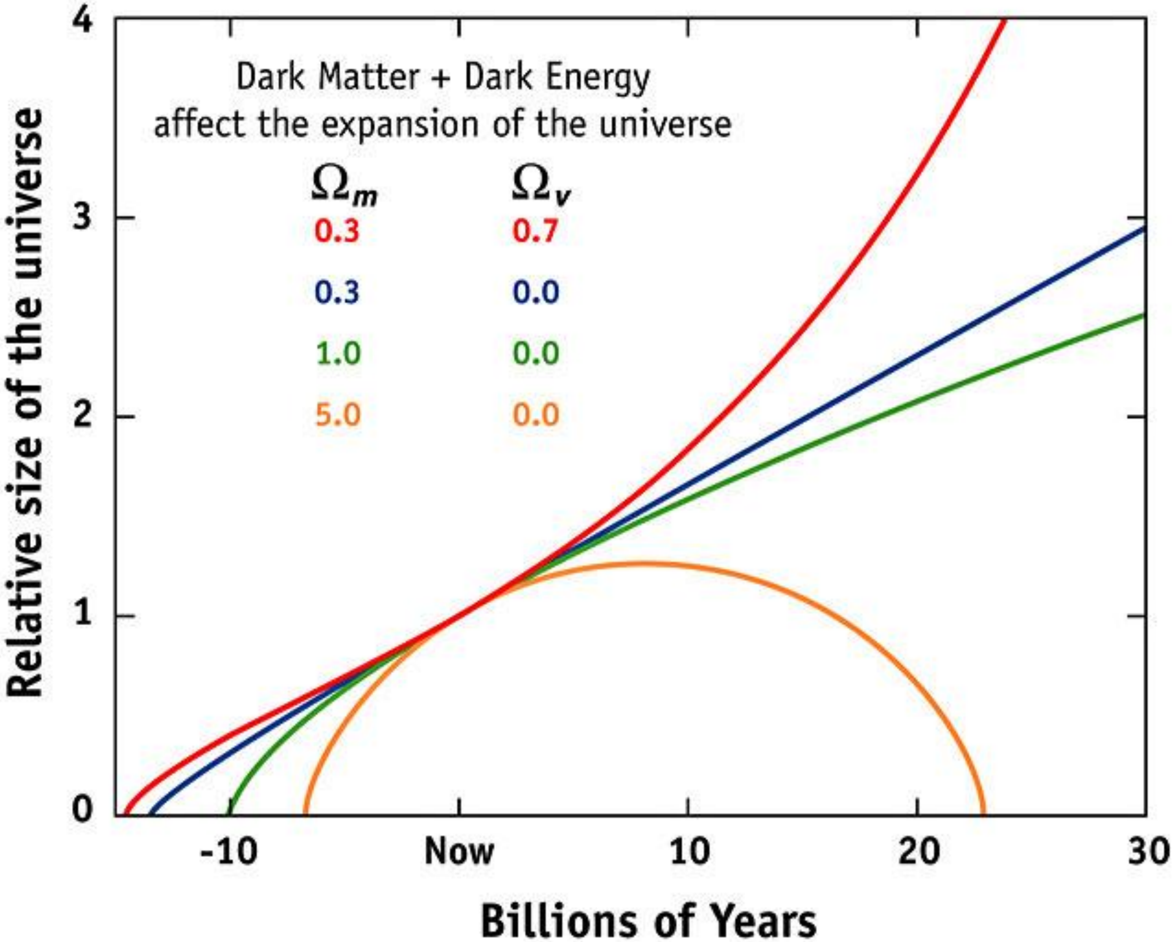
where $a_0 = 1$ and
$$H(z) = H_0 E(z) = H_0 \sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda)(1+z)^2}$$

- For nearby objects, z $\ll 1 \rightarrow d_c \approx \frac{cz}{H_0}$
  - Move according to Hubble's law, $v = H_0 d$, with $v = cz$

- The age of the universe is given by
$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{da}{\dot{a}} = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)E(z)}$$

For example: EdS $(K = 0, \Lambda = 0)$ has $E(z) = (1+z)^{3/2}$
$$\rightarrow t_0 = 2/(3H_0) \approx 6.7 h^{-1} Gyr$$

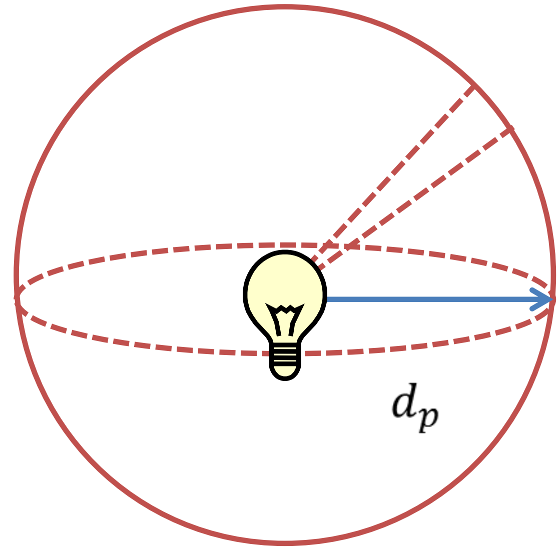
# Expansion of the Universe



- The luminosity distance is inferred from the inverse square law:

$$F = \frac{L}{4\pi {d_L}^2}$$

- Two effects as universe expands:
  - Individual photons lose energy $\propto (1+z)$
  - Photons arrive less frequently $\propto (1+z)$




- Received flux is thus

$$F = \frac{L}{4\pi {d_p}^2 (1+z)^2} \Rightarrow d_L = (1+z) d_p$$

$d_p$ is the transverse co-moving distance or proper motion distance

- For a flat universe, $d_p = d_c = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}$

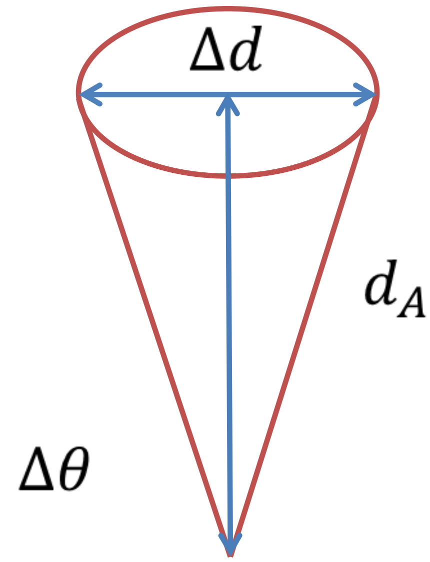
(otherwise it is very complicated – see Hogg, “Distance measures in cosmology”, astro-ph/9905116)

- The angular diameter distance comes from the relation between the observed angle and physical size of distant objects: $\Delta d = d_A \Delta\theta$

- The co-moving size of the object is $(1+z)\Delta d$, so

$$\Delta\theta = \frac{\Delta d}{d_A} = \frac{(1+z)\Delta d}{d_p}$$

$$\boxed{d_A = (1+z)^{-2} d_L = (1+z)^{-1} d_p}$$




- Supernova Cosmology measures $d_L(z)$, BAO and CMB measure $d_A(z)$

# Thermodynamics

- For a gas in thermal equilibrium: $(\hbar = c = k_B = 1)$

$$n = \frac{g}{(2\pi)^3}\int f(p) d^3p \qquad \text{(number density)}$$

$$\rho = \frac{g}{(2\pi)^3}\int f(p) E(p) d^3p \qquad \text{(energy density)}$$

$$p = \frac{g}{(2\pi)^3}\int f(p) \frac{|p|^2}{3E} d^3p \qquad \text{(pressure)}$$

$$f(p) = \frac{1}{\exp\left(\frac{E-\mu}{T}\right) \pm 1} \qquad \text{(distribution function)}$$

$\mu$: chemical potential

$g$: internal degrees of freedom

$+$: Fermions

$-$: Bosons

- Relativistic matter: $T >> m, T >> \mu$

$$\rho = \frac{\pi^2}{30} g T^4 \qquad \text{(Boson)}$$

$$\frac{7}{8}\frac{\pi^2}{30} g T^4 \qquad \text{(Fermion)}$$

$$p = \frac{1}{3}\rho, \quad n \propto T^3$$

- Non-relativistic matter: $m >> T$

$$\rho = mn$$

$$n = g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{m-\mu}{T}\right), p << \rho$$

- 2nd law of thermodynamics:
$$dS = \frac{1}{T}(d(\rho V) + pdV)$$

- When expansion timescale is long compared to reactions that maintain thermal equilibrium, gas undergoes adiabatic changes and entropy per co-moving volume $\mathrm{S(T)} = s(T)a^3 = const.$

- If $\mu = 0$, then $\frac{dp}{dT} = \frac{(\rho+p)}{T} = s(T)$ (conservation of energy)

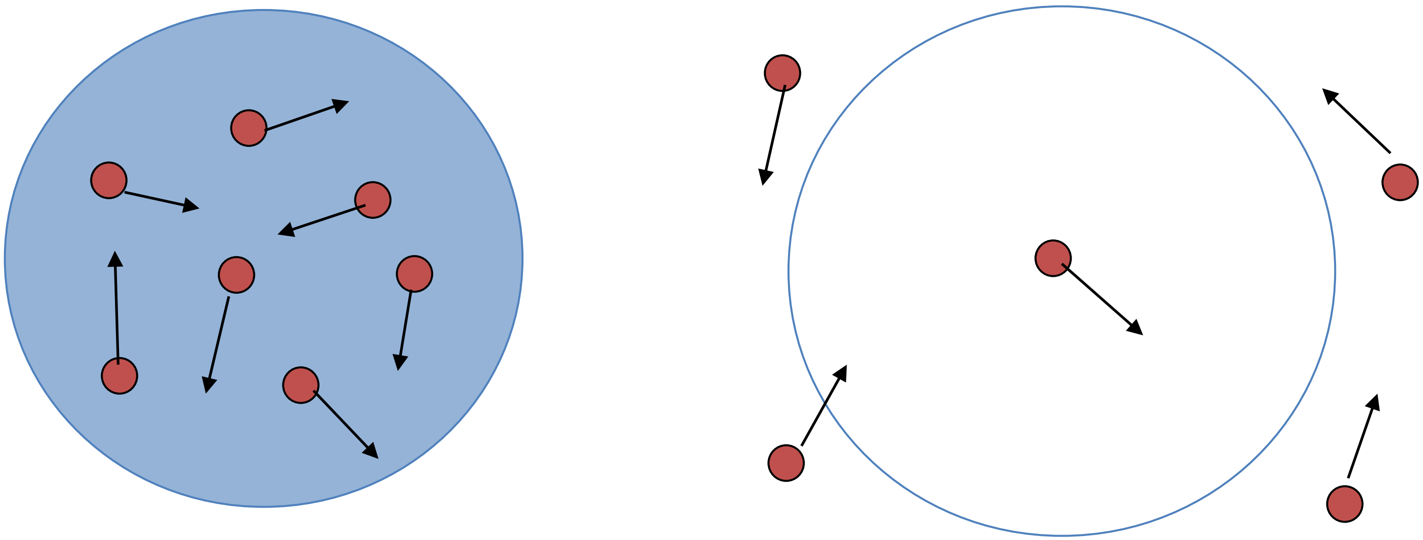
$$\Rightarrow \frac{a^3(\rho + p)}{T} = const. = \frac{a^3}{T}\left(\frac{4}{3}\rho\right)$$
(since $p = \rho/3$ for radiation)

- Stefan-Boltzmann law: $\rho \propto T^4 \rightarrow a^3T^3 = const. \Rightarrow \boxed{T \propto a^{-1}}$

*Temperature decreases as universe expands*

# Decoupling

- In expanding universe, particle can move at most $cH^{-1}$
- Interaction rate for some thermal process is $\Gamma[s^{-1}]$



- If interaction does not occur during Hubble time ($\Gamma^{-1} \gg H^{-1}$) the gas is no longer in thermal equilibrium

- Decoupling when $\Gamma < H$

# Freeze-out

- In thermal bath, a number of massive particles are exponentially suppressed

$$n = g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{m-\mu}{T}\right)$$

- If they decouple from the thermal bath, the number density is just diluted by expansion

$$n \propto a^{-3} \propto T^3$$

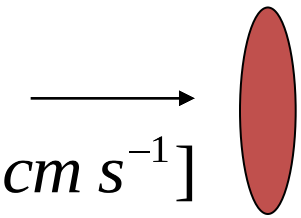
(precise treatment is given by Boltzmann eq.)

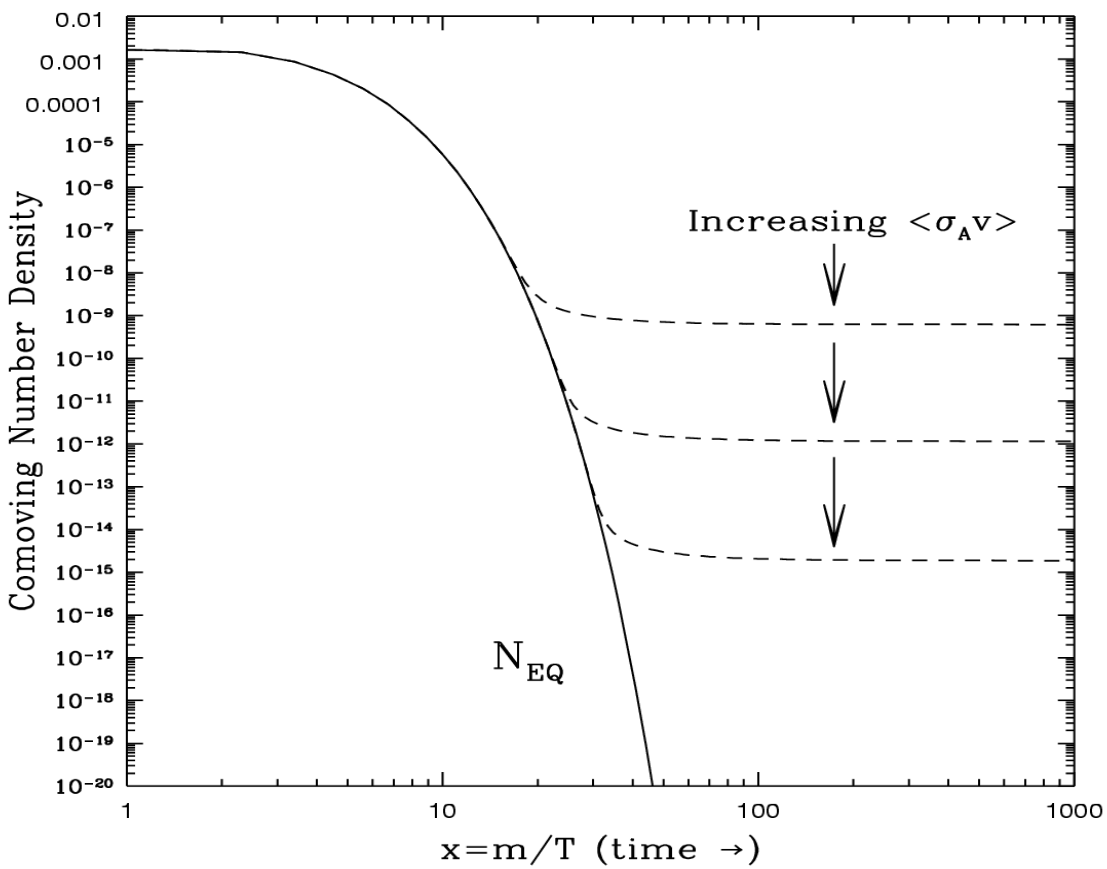
$$\frac{dn}{dt} = \underbrace{-3Hn}_{\textbf{COSMOLOGY}} - \underbrace{\langle \sigma v \rangle \left( n^2 - n_{eq}^2 \right)}_{\textbf{PARTICLE PHYSICS}}$$

$$\Gamma = \sigma v \, n_{eq}$$

$\sigma \, [cm^2]$

$v [cm \, s^{-1}]$






$\sigma$ : cross section

$v$ : average velocity

$n$ : number density

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