Cosmological perturbations

PhD lectures 2015

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Outline:

1. Metric perturbations and gauge transformations
2. Primordial perturbations from inflation
3. Adiabatic and isocurvature perturbations; non-Gaussianity

References:

• Malik and Wands, Phys Rep 475, 1 (2009); arXiv:0809.4944
• Kodama and Sasaki, Prog Theor Phys Supp 78, 1 (1984)
• Mukhanov, Feldman and Brandenberger, Phys Rep 215, 203 (1992)
• Lidsey et al, Rev Mod Phys 69, 1 (1997); astro-ph/9508078
• Bassett, Tsujikawa and Wands, Rev Mod Phys (2005); astro-ph/0507632

LECTURE 1:

DENSITY WAVES, METRIC PERTURBATIONS AND GAUGE TRANSFORMATIONS

Scalar perturbations

example:

fluid density, ρ (energy density ρc^2 / will set $c=1$)

velocity, $\vec{v}$
(neglect gravity!)

$$\dot{\rho} = -\vec{\nabla}\cdot(\rho\vec{v}) \quad \text{continuity}$$
$$(\rho\vec{v})^{\cdot} = -\vec{\nabla}P \quad \text{Euler}$$
$$P = P(\rho) \quad \text{barotropic equation of state}$$

simplest solution: "background" ρ = constant, $\vec{v} = \vec{0}$
(e.g. true vacuum $\rho = P = 0$)

next simplest case
small fluctuations about simple background
ρ_0 = constant
$\vec{v}_0 = \vec{0}$

consider $\rho = \rho_0 + \epsilon\delta\rho$
, $\vec{v} = \vec{0} + \epsilon\vec{u}$

where $\epsilon \ll 1$ is small parameter

substitute in to eqns. of motion

continuity: $\dot{\rho} = \epsilon\dot{\delta\rho} = -\vec{\nabla}\cdot[(\rho_0+\epsilon\delta\rho)\epsilon\vec{u}]$

Euler: $[(\rho_0+\epsilon\delta\rho)\epsilon\vec{u}]^{\cdot} = -\vec{\nabla}[P(\rho_0+\epsilon\delta\rho)]$
$= -\vec{\nabla}[P_0 + \epsilon\delta\rho P_0' + O(\epsilon^2)]$

linearise — only keep terms $\mathcal{O}(\epsilon^1)$
drop $\mathcal{O}(\epsilon^2)$

Continuity $\quad \epsilon\, \delta\dot{\rho} = -\epsilon \rho_0\, \vec{\nabla}\cdot\vec{u}$
Euler $\quad \rho_0\, \epsilon\, \dot{\vec{u}} = -\epsilon\, P_0' \vec{\nabla}\delta\rho$

} linear equations

combine $\Rightarrow \quad \delta\ddot{\rho} = -\rho_0\, \vec{\nabla}\cdot\dot{\vec{u}}$

$$\boxed{\delta\ddot{\rho} = P_0'\, \nabla^2\, \delta\rho}$$

e.g. 1-D $\quad \left(\frac{\partial^2}{\partial t^2} - P_0' \frac{\partial^2}{\partial x^2}\right)\delta\rho = 0$

plane wave soln: $\quad \delta\rho = A\cos(kx - \omega t + \phi)$

$$\omega^2 = c_s^2 k^2$$

where sound speed $c_s^2 = P_0' = \left(\frac{dP}{d\rho}\right)_{\rho_0}$

e.g. $\quad c_s^2 = 0$ for dust $P = 0$

$c_s^2 = \frac{1}{3}$ for radiation $P = \frac{1}{3}\rho c^2$ in 3D space

*

Fourier space ("k-space")

can decompose arbitrary ~~distribution~~ inhomogeneous function* ~~into eigenfun~~

$$\delta\varphi(t,\underline{x}) = \int \frac{d^3\underline{k}}{(2\pi)^3} \cdot \delta\varphi_{\underline{k}}(t) \cdot e^{i\underline{k}\cdot\underline{x}}$$

$$\Leftrightarrow \quad \delta\varphi_{\underline{k}}(t) = \int d^3\underline{x} \; \delta\varphi(t,\underline{x}) \cdot e^{-i\underline{k}\cdot\underline{x}}$$

note :- $\delta\varphi$ real $\Rightarrow$ $\delta\varphi_{\underline{k}} = \delta\varphi_{-k}^{*}$

- eigenfunctions of spatial Laplacian

$$\partial^2\left(\delta\varphi_{\underline{k}}(t)\cdot e^{i\underline{k}\cdot\underline{x}}\right) = -\delta^{ij} k_i k_j \left(\quad \right)$$

$$= -k^2\left(\delta\varphi_{\underline{k}}(t)\cdot e^{i\underline{k}\cdot\underline{x}}\right)$$

- * complete orthonormal basis

$$\int d^3\underline{x} \; \frac{e^{i\underline{k}\cdot\underline{x}} \cdot e^{i\underline{k}'\cdot\underline{x}}}{(2\pi)^3}$$

$$= \delta^{(3)}(\underline{k}+\underline{k}')$$

Dirac δ-function

Statistical properties:

~~Power spectra~~

describe ~~properties of~~ distribution / ensemble
(not just one realisation)

power spectrum

$$\langle \delta\varphi^*_{\underline{k}} \; \delta\varphi_{\underline{k}'} \rangle \doteq (2\pi)^3 \cdot P_{\delta\varphi}(k) \cdot \delta^{(3)}(\underline{k} + \underline{k}')$$

↑ ensemble average ↑ consequence of spatial homog.

bispectrum

$$\langle \delta\varphi_{\underline{k}} \; \delta\varphi_{\underline{k}'} \; \delta\varphi_{\underline{k}''} \rangle = (2\pi)^3 \cdot B_{\delta\varphi}(\underline{k}, \underline{k}', \underline{k}'') \; \delta^{(3)}(\underline{k} + \underline{k}' + \underline{k}'')$$

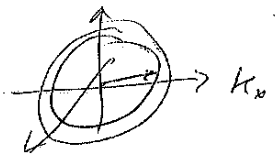
trispectrum, etc

note :- in real space

variance

$$\langle \varphi^2(x) \rangle = \int \frac{d^3\underline{k} \, d^3\underline{k}'}{(2\pi)^6} \langle \varphi_{\underline{k}} \varphi_{\underline{k}'} \rangle \, e^{i(\underline{k}+\underline{k}')\underline{x}}$$

$$= \int \frac{d^3\underline{k}}{(2\pi)^3} \, P_{\delta\varphi}(k)$$



$$= \int \frac{4\pi k^3}{(2\pi)^3} \cdot P_{\delta\varphi}(k) \cdot d(\ln k)$$

$$= \int \mathcal{P}_{\delta\varphi}(k) \cdot d(\ln k)$$

where dimensionless power spectrum

$$\mathcal{P}_{\delta\varphi}(k) = \frac{4\pi k^3}{(2\pi)^3} \cdot P_{\delta\varphi}(k)$$

- for <u>Gaussian perturbations</u> $B_{\delta\varphi} = 0$
  and for all odd ~~correlations~~ moments ($\langle \delta\varphi^5 \rangle = 0$, etc)

- take first-order perturbations to be Gaussian
  ⟹ higher-order pertbns describe non-Gaussianity

Vector perturbations

decompose arbitrary 3-vector $\vec{V}$
into scalar (potential) part
and "vector" (divergence-free) part

$$\vec{V} = \vec{\nabla} V^{(s)} + \vec{V}^{(v)}$$

where $\vec{\nabla} \cdot \vec{V}^{(v)} = 0$

Fourier basis:

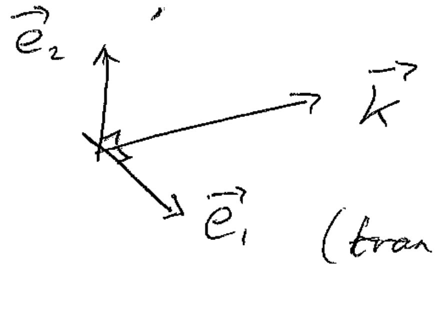
$$V^{(s)}(t,x) = \int \frac{d^3k}{(2\pi)^3} V_k^{(s)}(t)\, e^{i\vec{k}.\vec{x}}$$

$$\vec{V}^{(v)}(t,x) = \int \frac{d^3k}{(2\pi)^3} \left\{ V_k^{(v)}(t)\, \vec{e}_1(\vec{k}) + \bar{V}_k^{(v)}(t)\, \vec{e}_2(\vec{k}) \right\} e^{i\vec{k}.\vec{x}}$$

where $\vec{e}_1$ and $\vec{e}_2$ are orthonormal transverse basis

$$\vec{e}_1 \cdot \vec{e}_2 = 0 = \vec{e}_1 \cdot \vec{k} = \vec{e}_2 \cdot \vec{k}$$

$$\vec{e}_1 \cdot \vec{e}_1 = 1 = \vec{e}_2 \cdot \vec{e}_2$$



(transverse) : each wave mode $V_k^{(v)} \vec{e}_1 e^{i\vec{k}.\vec{x}}$ is divergence-free

$$\vec{\nabla} \cdot \left(V_k^v \vec{e}_1 e^{i\vec{k}.\vec{x}} \right) = i V_k^v (\vec{k} \cdot \vec{e}_1)^{\,0} e^{i\vec{k}.\vec{x}} = 0$$

Tensor perturbations:

decompose symmetric tensor T_{ij} into scalar, vector & "tensor" (transverse, + trace-free)

$$T_{ij} = \vec{\nabla}_i \vec{\nabla}_j (S) + \vec{\nabla}_i \vec{V}_j + \vec{\nabla}_j \vec{V}_i + h_{ij}$$

where transverse $\vec{\nabla}^i h_{ij} = 0$

tracefree $\delta^{ij} h_{ij} = 0$

$$h_{ij} = \int \frac{d^3k}{(2\pi)^3} \left\{ h_k(t)\, q_{ij}(\vec{k}) + \bar{h}_k(t)\, \bar{q}_{ij}(\vec{k}) \right\} e^{i\vec{k}\cdot\vec{x}}$$

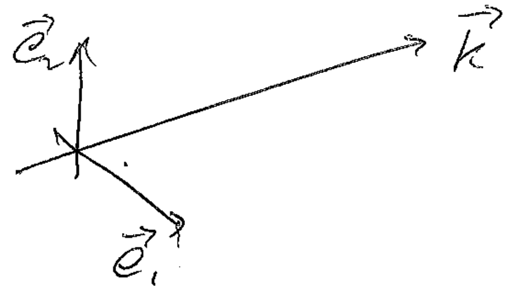
where two polarisation

$$q_{ij} = \frac{1}{\sqrt{2}} \left[\vec{e}_{1i} \vec{e}_{1j} - \vec{e}_{2i} \vec{e}_{2j} \right]$$

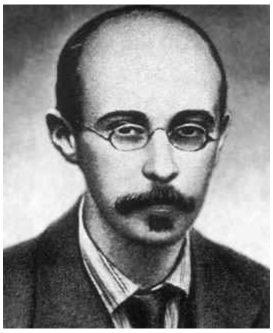
$$\bar{q}_{ij} = \frac{1}{\sqrt{2}} \left[\vec{e}_{1i} \vec{e}_{2j} + \vec{e}_{1j} \vec{e}_{2i} \right]$$

transverse $k_i q_{ij} = 0 = k_i \bar{q}_{ij}$

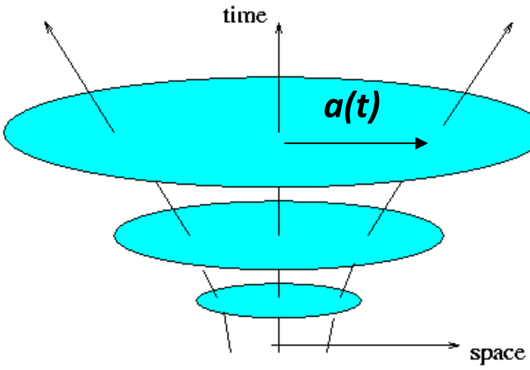
+ tracefree



Homogeneous FRW background



- Unique *time-slicing* (foliation) of 4D spacetime into homogeneous 3D space, with homogeneous matter density, $\rho(t)$



scale factor

$$ds^2 = a^2(t)\, dx^2$$

- Comoving world lines define natural choice of *threading* (spatial coordinates)

cosmic fluid equations:

energy density ρ

pressure P

momentum density $\vec{q} = (\rho + P)\vec{v}$

$$\dot{\rho} = -\vec{\nabla}.\vec{q} - 3H\rho$$

$$\dot{\vec{q}} = -\vec{\nabla}p - 3H\vec{q}$$

Hubble expansion, H, dilutes density and momentum

eliminate pressure $\vec{\nabla}P = c_s^2\vec{\nabla}\rho$

where c_s^2 = adiabatic sound speed = 1/3 in hot big bang

to obtain second-order wave equation

$$\ddot{\rho} + 3H\dot{\rho} = c_s^2\nabla^2\rho$$

cosmic sound waves:

wave equation in an expanding spacetime:

$$\frac{d^2}{dt^2}\rho + 3H\frac{d}{dt}\rho = c_s^2\frac{d^2}{dx^2}\rho$$

Characteristic timescales for wavelength λ

- *oscillation period/wavelength* λ / c_s
- *Hubble damping time-scale* H^{-1}

- *small-scales* $\lambda < c_s H^{-1}$ *under-damped oscillator*
- *large-scales* $\lambda > c_s H^{-1}$ *over-damped ("frozen-in")*

it isn’t always quite that simple...

in general relativity should consider both

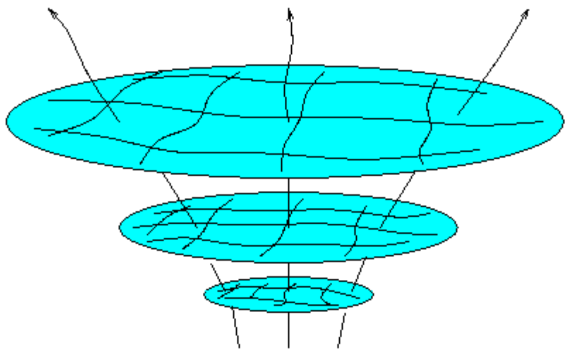
- effect of curved spacetime on matter
- effect of matter on curved spacetime

...leads to non-linear coupled equations

but often the simple picture works for linearised perturbations

Inhomogeneous perturbations

- No unique *time-slicing* (foliation) of 4D spacetime 3D space
  inhomogeneous matter density, $\rho(t,x)$



- Comoving world lines may define one choice of *threading*
  but not unique

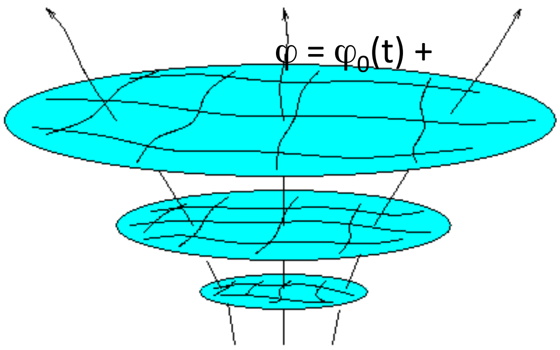
  free to choose of coordinates *(gauge problem)*
  - not a bug, but a feature of general relativity!

Inhomogeneous perturbations on homogeneous background

Homogeneous model: $\varphi = \varphi_0(t)$

Inhomogeneous model: $\varphi = \varphi(t,x)$
$= \varphi_0(t) + \Delta\varphi(t,x)$
$= \varphi_0(t) + \varepsilon\, \delta_1\varphi(t,x) + (1/2)\, \varepsilon^2\, \delta_2\varphi(t,x) + \ldots$

typically for linear perturbations only I will write $\varphi = \varphi_0(t) + \delta\varphi(t,x) + \ldots$



But in an inhomogeneous spacetime
- no obvious, preferred time slicing
  $T = t + \delta t(t,x) + \ldots$
- no obvious, preferred spatial threading
  $X = x + \delta x(t,x) + \ldots$

T and t, or X and x, only need to agree at zeroth order (same background)

This arbitrariness in the choice of coodinates is the *gauge problem*

Inhomogeneous perturbations on homogeneous background

Homogeneous model: $\varphi = \varphi_0(t)$

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$= \varphi_0(t) + \Delta\varphi(t,x)$
$= \varphi_0(t) + \varepsilon\, \delta_1\varphi(t,x) + (1/2)\, \varepsilon^2\, \delta_2\varphi(t,x) + \ldots$

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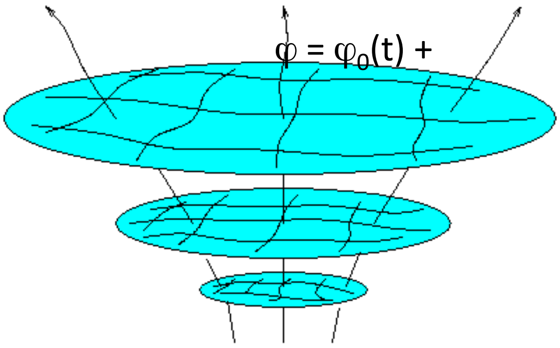

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FRW metric

Background line element:

$$ds^2 = a^2(\eta)\,[\,-d\eta^2 + \gamma_{ij}\,dx^i\,dx^j\,]$$

where
- η is conformal time coordinate
- γ_{ij} is spatial metric on maximally symmetric 3-space, curvature K

$$g^0_{\mu\nu} = a^2(\eta)\begin{pmatrix} -1 & 0 \\ 0 & \gamma_{ij} \end{pmatrix}$$

Metric perturbations

using notation of Mukhanov, Feldman and Brandenberger (1992)

Linear perturbation: $$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}$$

schematically:

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -(1+2\phi) & B_{|i} - S_i \\ B_{|j} - S_j & (1-2\psi)\gamma_{ij} + 2E_{|ij} + F_{i|j} + F_{j|i} + h_{ij} \end{pmatrix}$$

hence the perturbed line element is

$$ds^2 = a^2(\eta)\left\{-(1+2\phi)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left[(1-2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right]dx^i dx^j\right\}$$

4 scalars:	ϕ, B, ψ, E
2 transverse vectors:	S_i, F_i
1 transverse, tracefree tensor:	h_{ij}

Gauge transformation of metric perturbations

Coordinate transformation:

$$\tilde{\eta} = \eta + \xi^0\,, \qquad \tilde{x}^i = x^i + \xi_|{}^i + \bar{\xi}^i\,,$$

where infinitesimal

$$d\xi^0 = \xi^{0\prime} d\tilde{\eta} + \xi^0{}_{|i} d\tilde{x}^i$$
$$d\xi = \xi' d\tilde{\eta} + \xi{}_{|j} d\tilde{x}^j$$
$$d\bar{\xi}^i = \bar{\xi}^{i\prime} d\tilde{\eta} + \bar{\xi}^i{}_{|j} d\tilde{x}^j$$

implies transformed coordinate infinitesimals:

$$d\eta = d\tilde{\eta} - \xi^{0\prime} d\tilde{\eta} - \xi^0{}_{|i} d\tilde{x}^i$$
$$dx^i = d\tilde{x}^i - (\xi'_|{}^i + \bar{\xi}^{i\prime}) d\tilde{\eta} - (\xi_|{}^i{}_j + \bar{\xi}^i{}_{|j}) d\tilde{x}^j$$

which, along with $a(\eta) = a(\tilde{\eta}) - \xi^0 a(\tilde{\eta})'$,
substituted in the line element, gives:

$$\begin{aligned} ds^2 \;=\; & a^2(\tilde{\eta}) \Big\{ -\left(1 + 2\left(\phi - h\xi^0 - \xi^{0\prime}\right)\right) d\tilde{\eta}^2 + 2\left(B + \xi^0 - \xi'\right)_{|i} d\tilde{\eta} d\tilde{x}^i - 2\left(S_i + \bar{\xi}'_i\right) d\tilde{\eta} d\tilde{x}^i \\ & + \Big[\left(1 - 2\left(\psi + h\xi^0\right)\right)\gamma_{ij} + 2\left(E - \xi\right)_{|ij} + 2\left(F_{i|j} - \bar{\xi}_{i|j}\right) + h_{ij}\Big] d\tilde{x}^i d\tilde{x}^j \Big\}\,. \end{aligned}$$

Gauge transformation rules for metric perturbations

Can be written in same form as the original perturbed line element, but in terms of the new coordinates:

$$ds^2 = a^2(\tilde{\eta})\left\{-(1+2\tilde{\phi})d\tilde{\eta}^2 + 2(\tilde{B}_{|i} - \tilde{S}_i)d\tilde{\eta}d\tilde{x}^i + \left[(1-2\tilde{\psi})\gamma_{ij} + 2\tilde{E}_{|ij} + 2\tilde{F}_{i|j} + \tilde{h}_{ij}\right]d\tilde{x}^i d\tilde{x}^j\right\}$$

where

- scalars:

$$\begin{aligned} \tilde{\phi} &= \phi - h\xi^0 - \xi^{0\prime} \\ \tilde{\psi} &= \psi + h\xi^0 \\ \tilde{B} &= B + \xi^0 - \xi' \\ \tilde{E} &= E - \xi \end{aligned}$$

- Vectors:

$$\begin{aligned} \tilde{F}_i &= F_i - \bar{\xi}_i, \\ \tilde{S}_i &= S_i + \bar{\xi}'_i. \end{aligned}$$

- Tensors:

$$h_{ij} = h_{ij}$$

First-order field equations

Einstein equations:
Two constraint equations:

$$3h(\psi' + h\phi) + (k^2 - 3\mathcal{K})\psi + hk^2\sigma = -4\pi G a^2 \delta\rho$$ (energy density)

$$\psi' + h\phi + \mathcal{K}\sigma = -4\pi G a^2 (\rho + p)(v + B)$$ (momentum)

Two evolution equations:

$$\psi'' + 2h\psi' - \mathcal{K}\psi + h\phi' + (2h' + h^2)\phi = 4\pi G a^2 \left(\delta p - \frac{2}{3}k^2 a^2 \Pi\right)$$ (curvature)

$$\sigma' + 2h\sigma - \phi + \psi = 8\pi G a^2 \Pi$$ (shear)

Energy-momentum conservation:

$$\delta\rho'_\alpha + 3h(\delta\rho_\alpha + \delta p_\alpha) = (\rho_\alpha + p_\alpha)\left[k^2(v_\alpha + E') + 3\psi'\right]$$ (energy density)

$$[(\rho_\alpha + p_\alpha)(v_\alpha + B)]' = -(\rho_\alpha + p_\alpha)\left[4h(v_\alpha + B) + \phi\right] - \delta p_\alpha + \frac{2}{3}(k^2 - 3\mathcal{K})\Pi_\alpha$$

(momentum)