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Cosmological perturbations

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Outline:

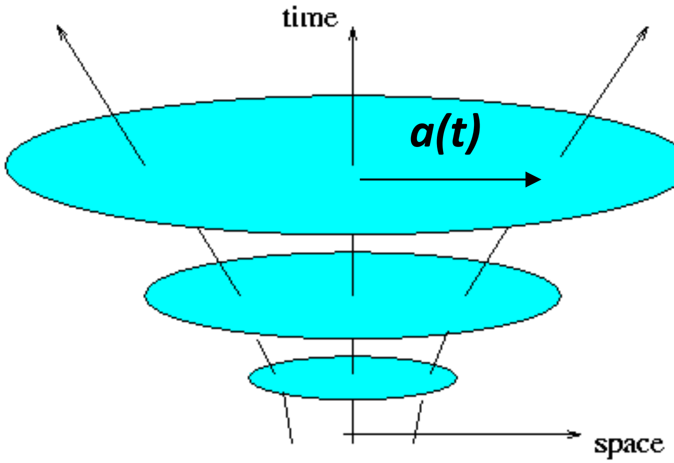
1. Primordial perturbations from inflation
2. Metric perturbations and gauge transformations
3. Field equations; adiabatic and isocurvature perturbations; non-Gaussianity

References:

- Bardeen, Phys Rev D22, 1882 (1980)
- Kodama and Sasaki, Prog Theor Phys Supp 78, 1 (1984)
- Mukhanov, Feldman and Brandenberger, Phys Rep 215, 203 (1992)
- Lidsey et al, Rev Mod Phys 69, 1 (1997); astro-ph/9508078
- Bassett, Tsujikawa and Wands, Rev Mod Phys (2005); astro-ph/0507632
- Malik and Wands, Phys Rep 475, 1 (2009), arXiv:0809:4944

Homogeneous FRW background

- Obvious *slicing* (foliation) of 4D spacetime into homogeneous 3D space, with homogeneous matter density, $\rho(t)$



scale factor

$$ds^2 = a^2(t)\gamma_{ij}dx^i dx^j$$

- Comoving world lines define natural choice of *threading* (spatial coordinates)

Inhomogeneous linear perturbations about homogeneous background

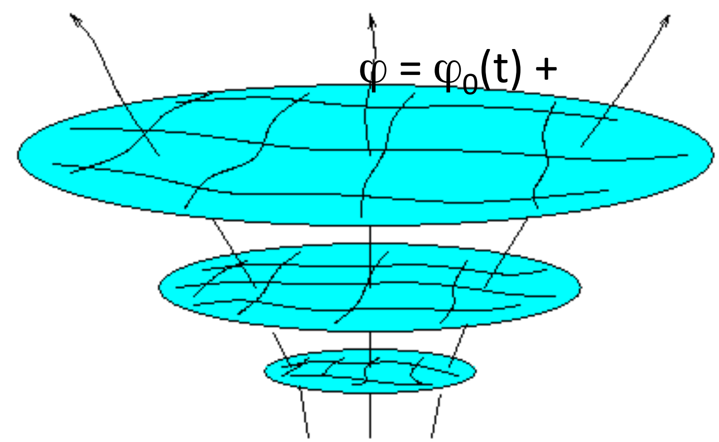
Homogeneous model: $\varphi = \varphi_0(t)$

Inhomogeneous model: $\varphi = \varphi(t,x)$

$= \varphi_0(t) + \Delta\varphi(t,x)$

$= \varphi_0(t) + \varepsilon\, \delta_1\varphi(t,x) + (1/2)\, \varepsilon^2\, \delta_2\varphi(t,x) + \ldots$

typically for linear perturbations only I will write $\varphi = \varphi_0(t) + \delta\varphi(t,x) + \ldots$



But in an inhomogeneous spacetime

- no obvious, preferred time slicing

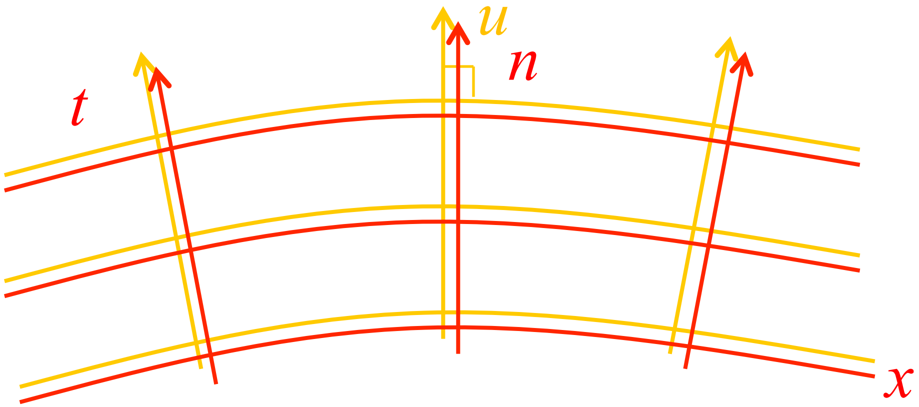
  $t' = t + \delta t(t,x) + \ldots$

- no obvious, preferred spatial threading

  $x' = x + \delta x(t,x) + \ldots$

t and t', or x and x', only need to agree at zeroth order (same background)

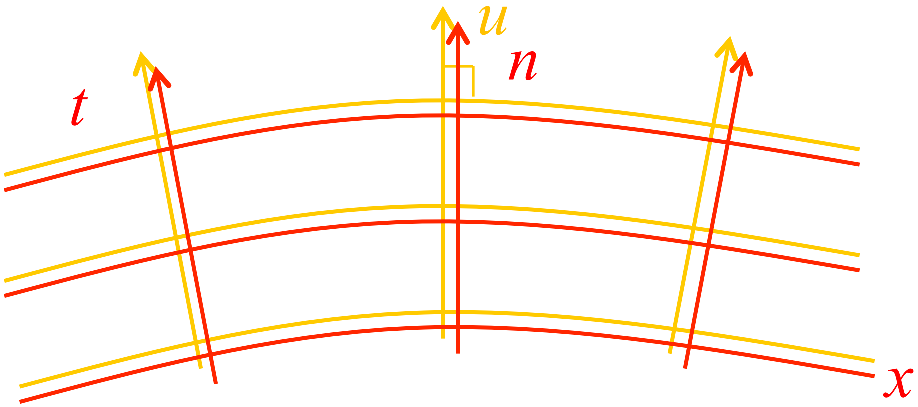
This arbitrariness in the choice of coodinates is the *gauge problem*



FRW cosmology

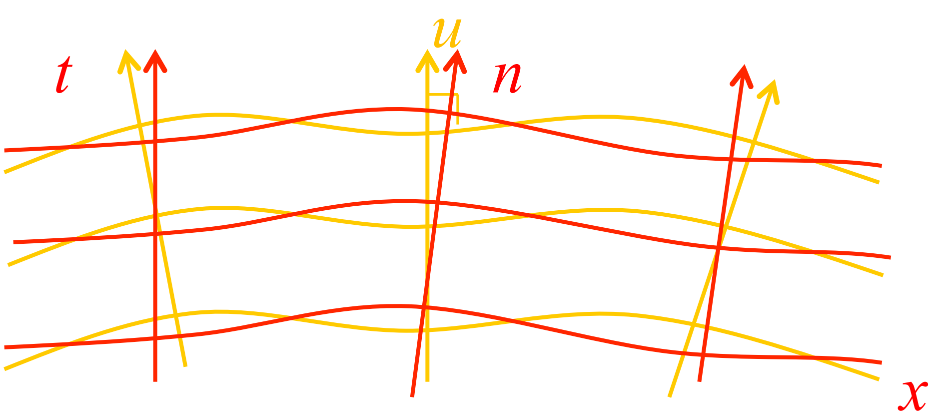
preferred coordinates for homogeneous and isotropic space

preferred space+time split in FRW cosmology breaks symmetry of Einstein's theory



FRW cosmology

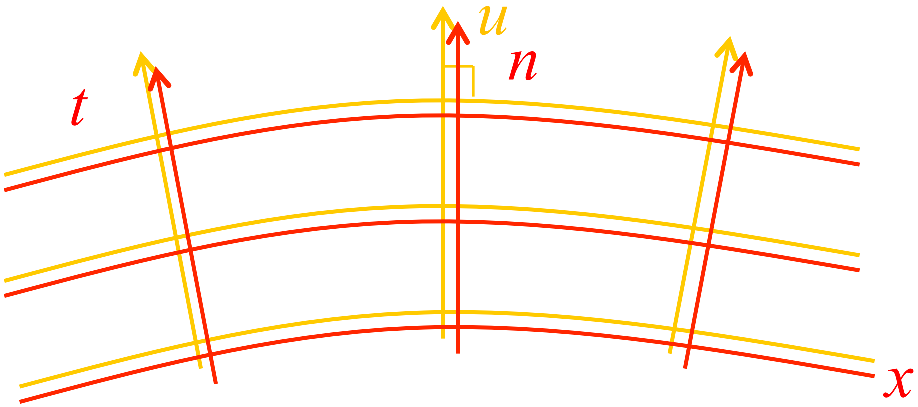
no unique choice of time (slicing) and space coordinates (threading) in an inhomogeneous universe



FRW cosmology
+ perturbations

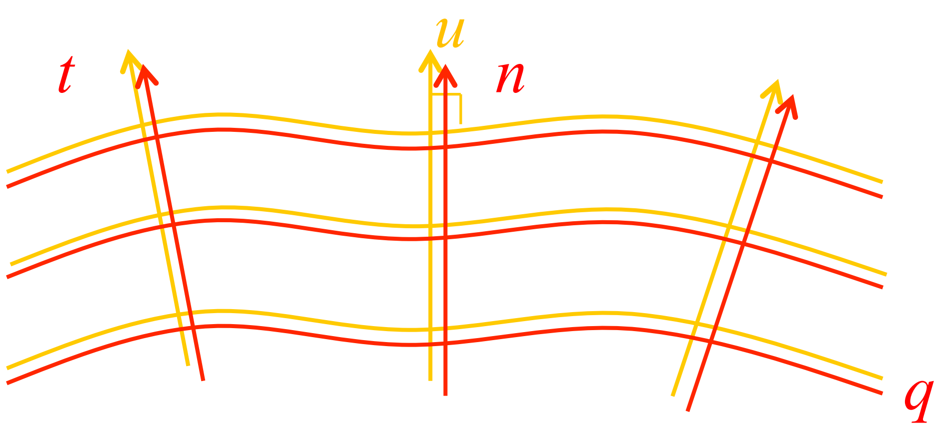
arbitrary gauge (t,x)

gauge problem: find different perturbations in different gauges



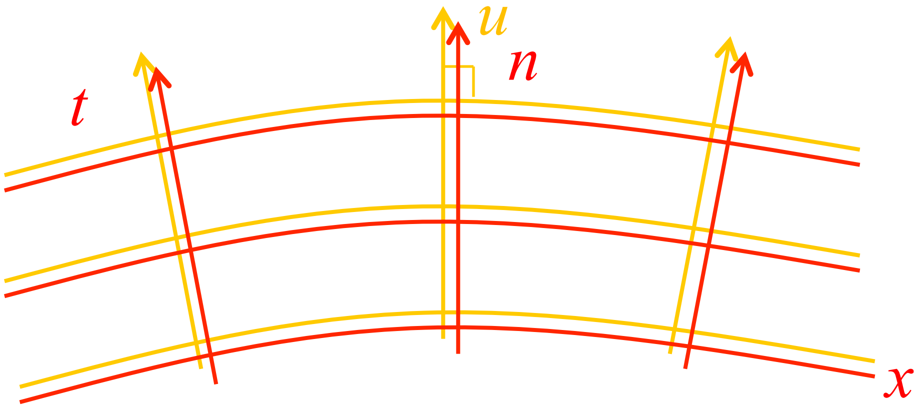
FRW cosmology

synchronous+comoving with pressureless cold dark matter
time-slicing orthogonal to comoving worldlines



FRW cosmology
+ perturbations

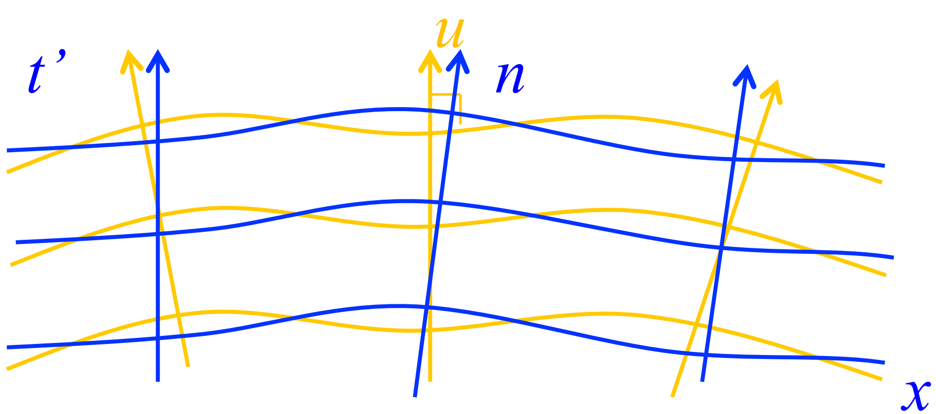
comoving-Lagrangian coordinates ***(t,q)***



FRW cosmology

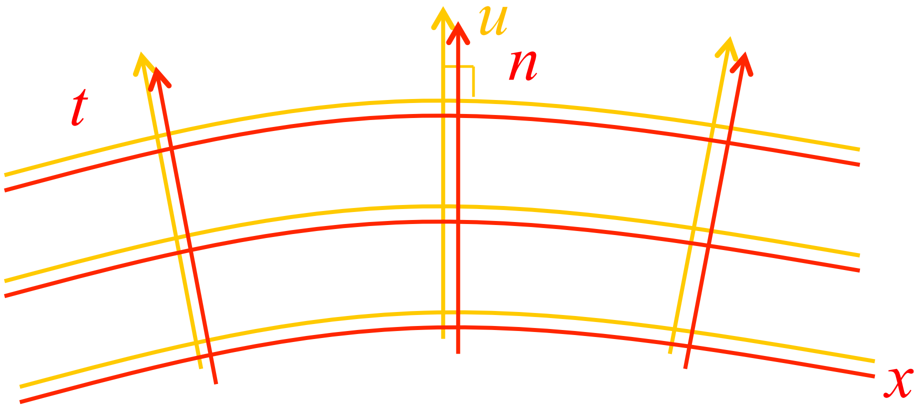
Poisson = conformal Newtonian = longitudinal gauge

hypersurface-orthogonal 4-vector field ***n*** *is shear-free*



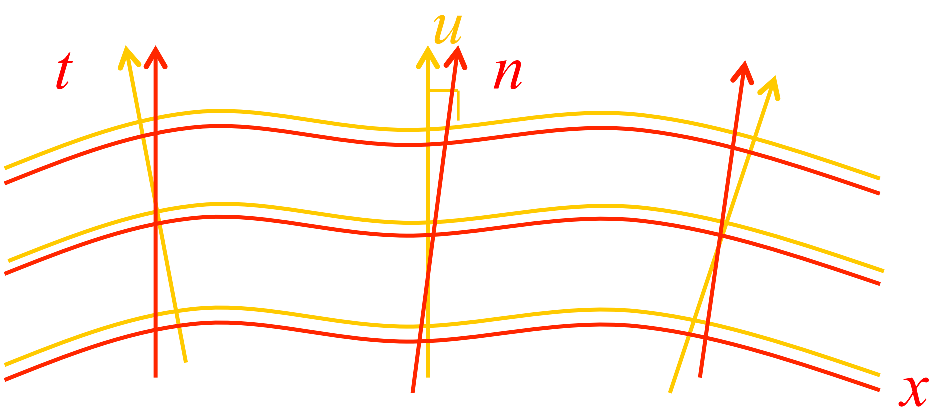
FRW cosmology
+ perturbations

Poisson gauge coordinates ***(t',x)***



FRW cosmology

time-slicing orthogonal to comoving worldlines
spatial threading is same as Poisson gauge (Eulerian, not Lagrangian)



FRW cosmology
+ perturbations

total-matter coordinates *(t,x)*

Standard Newtonian+Gaussian initial fields

Gaussian primordial metric fluctuations $\zeta(x)$ from inflation + linear Einstein-Boltzmann code (e.g., CMBfast, CAMB, CLASS)

Gaussian initial Newtonian potential $\Phi = (3/5)\zeta$

Gaussian initial matter density using Poisson equation

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\delta$$

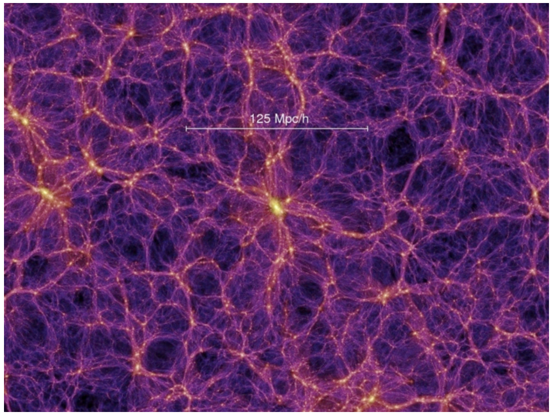
Gaussian initial displacement $\vec{\nabla}.\vec{\Psi} = -\delta$

Newtonian N-body simulations

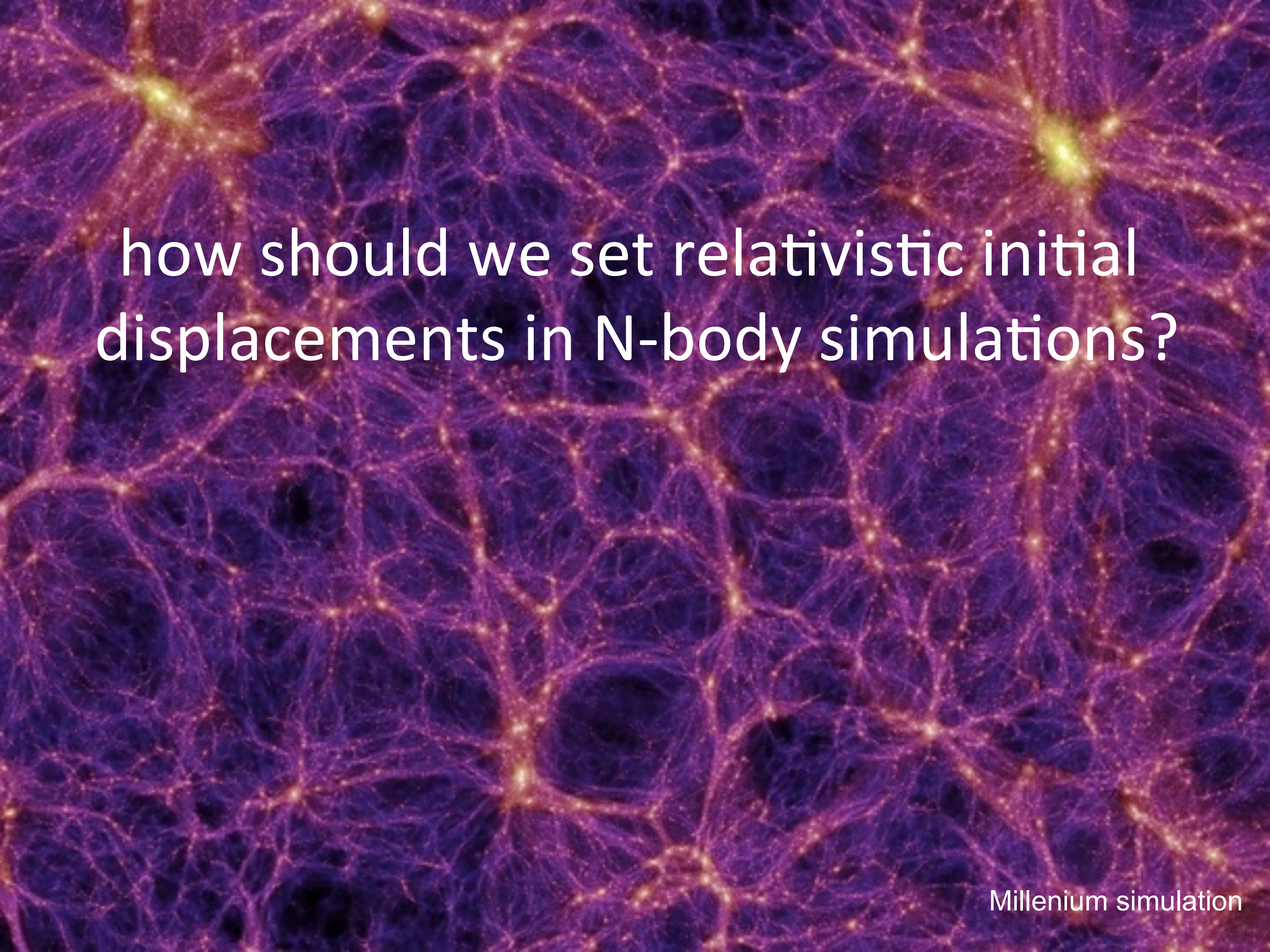
$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\delta$$

$$\dot{\delta} + \vec{\nabla}.((1+\delta))\vec{v} = 0$$

$$\dot{\vec{v}} + \mathcal{H}\vec{v} + (\vec{v}.\vec{\nabla})\vec{v} = -\vec{\nabla}\Phi$$



how should we set relativistic initial displacements in N-body simulations?



Millenium simulation

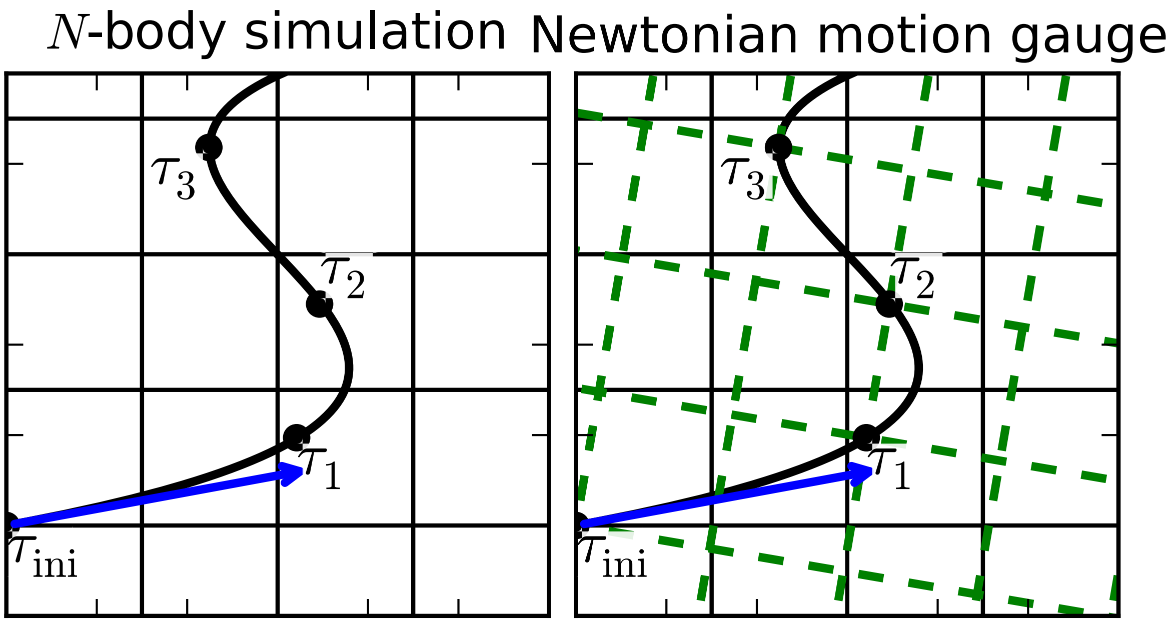
Newtonian motion gauges

Fidler, Rampf, Tram, Crittenden, Koyama & Wands, arXi:1606.05588

- in GR coordinates are arbitrary
- coordinates+metric define the physical spacetime
  - so… construct gauge such that

    Newtonian displacement = GR displacement

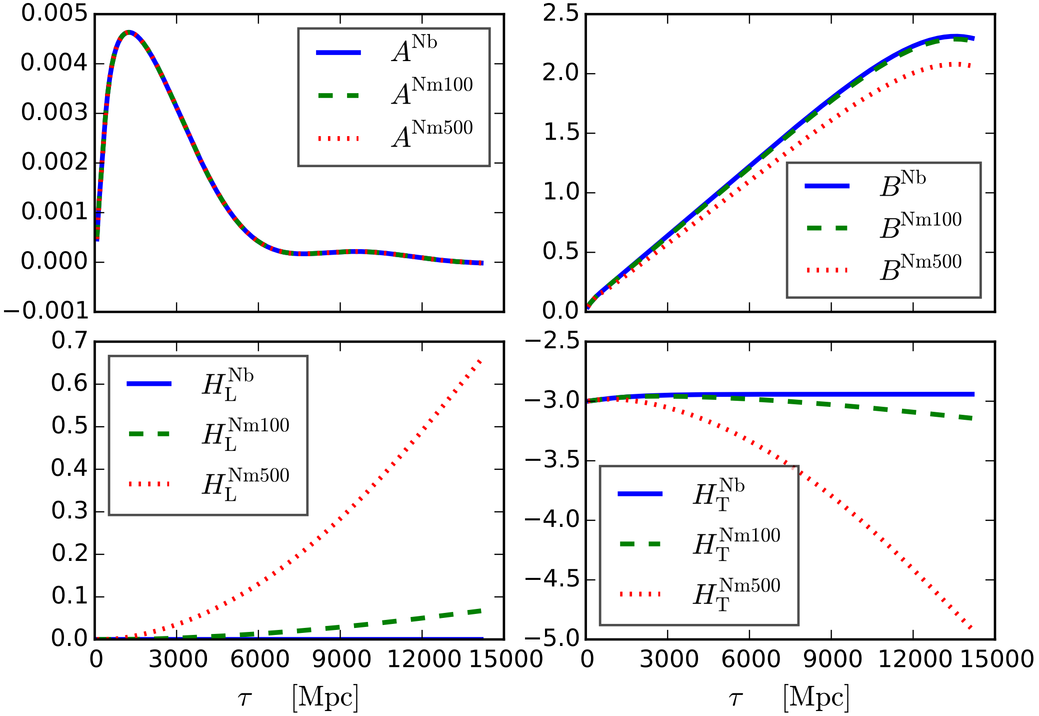
$$\Psi^{N}(t, \vec{q}) \equiv \Psi^{GR}(t, \vec{q})$$



Newtonian motion gauges

Fidler, Rampf, Tram, Crittenden, Koyama & Wands, arXi:1606.05588

- **N-body simulations** solve Newtonian non-linear collapse for matter
- **Einstein-Boltzmann code** solves relativistic perturbation equations for metric potentials and radiation in Newtonian motion gauge



$$g_{00} = -a^2(1+2A)\,,$$

$$g_{0i} = -a^2 B_i\,,$$

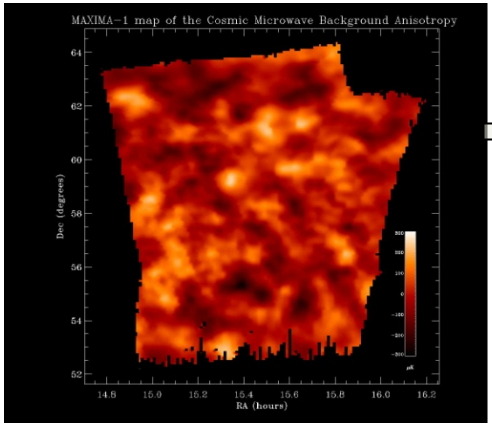
$$g_{ij} = a^2\left[\delta_{ij}\left(1+2H_{\mathrm{L}}\right) - 2H_{\mathrm{T}\,ij}\right]$$

Primordial perturbations from inflation

Standard model of structure formation

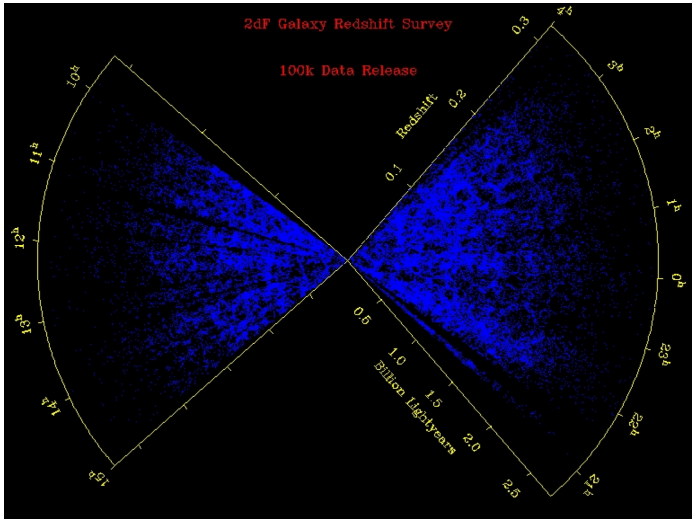
primordial perturbations

in cosmic microwave background



gravitational

instability



large-scale structure of our Universe

new observational data offers precision tests of

- cosmological parameters
- the nature of the primordial perturbations

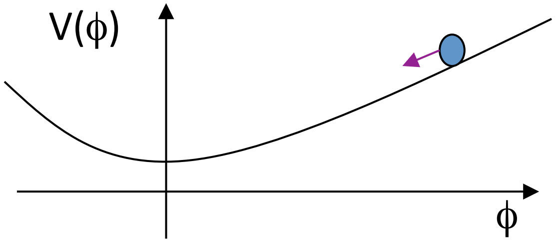
Inflation: initial false vacuum state drives accelerated expansion
zero-point fluctuations yield spectrum of perturbations

Cosmological inflation:

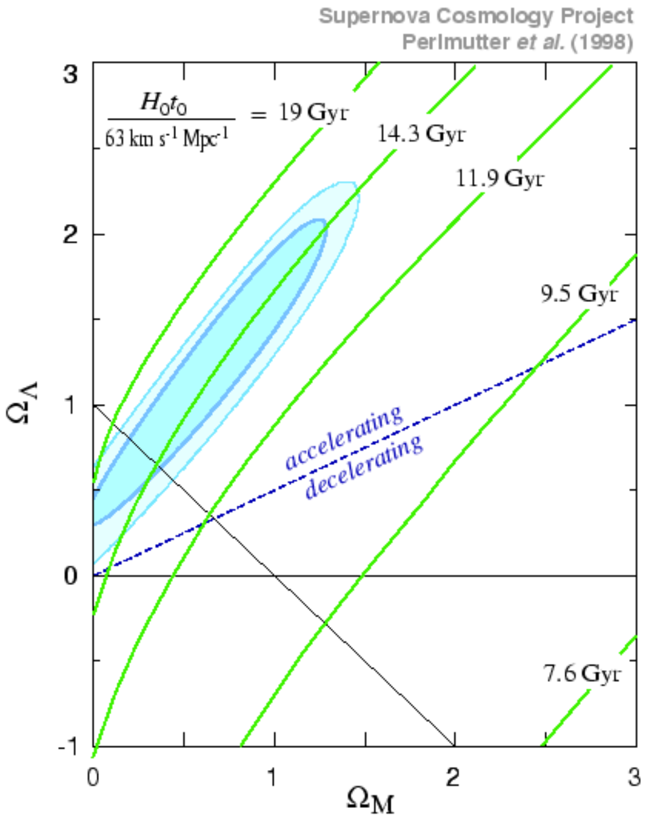
Starobinsky (1980)
Guth (1981)

- period of accelerated expansion
  in the very early universe
- requires negative pressure

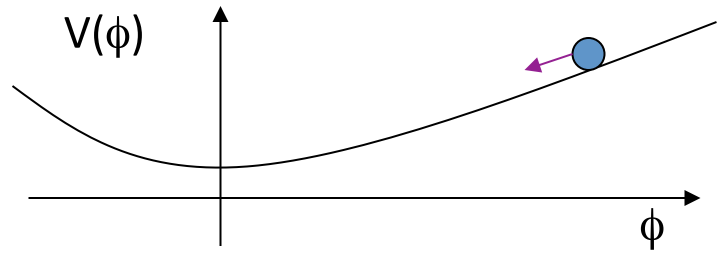
e.g. self-interacting scalar field



- speculative and uncertain physics
- just the kind of peculiar cosmological behaviour we observe today



Single-field inflation:



classical evolution equations

- *Klein-Gordon:* $$\cancel{\ddot{\phi}} + 3H\,\dot{\phi} = \frac{dV}{d\phi}$$
- *Hubble expansion rate:* $$H^2 = \frac{8\pi G}{3}\left(\cancel{\frac{1}{2}\dot{\phi}^2} + V(\phi)\right)$$

accelerated expansion, $\ddot{a} > 0$*, for* $V(\phi) > \dot{\phi}^2$

slow-roll solution for potential-dominated, over-damped evolution gives useful approximation to growing mode for { ε , |η| } << 1

where slow-roll parameters:

$$\varepsilon \equiv \frac{M_P^{\,2}}{16\pi}\left(\frac{V_\phi}{V}\right)^2 \qquad \eta \equiv \frac{M_P^{\,2}}{8\pi}\left(\frac{V_{\phi\phi}}{V}\right) = \frac{m^2}{H^2}$$

Perturbations in FRW universe:

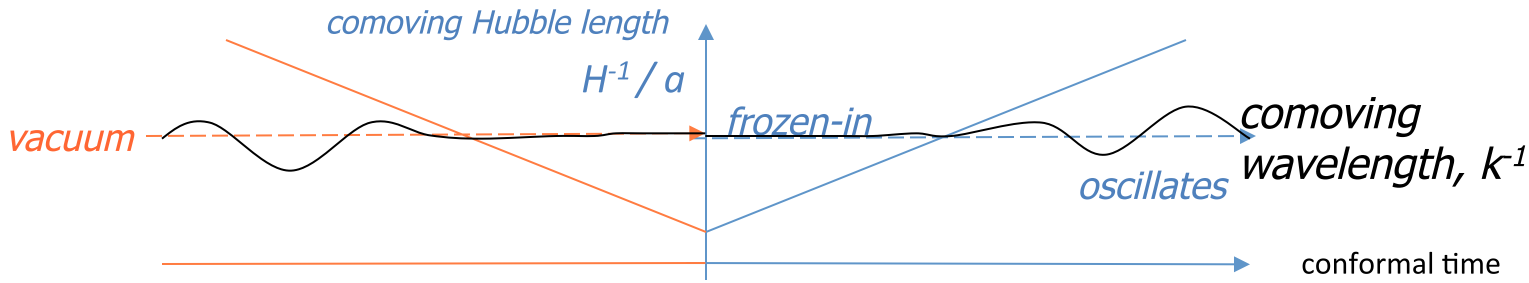
wave equation

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \nabla^2\delta\phi = 0$$

Characteristic timescales for comoving wavemode k

- *oscillation period/wavelength* a / k
- *Hubble damping time-scale* H^{-1}

- *small-scales k > aH* *under-damped oscillator*
- *large-scales k < aH* *over-damped oscillator*



inflation

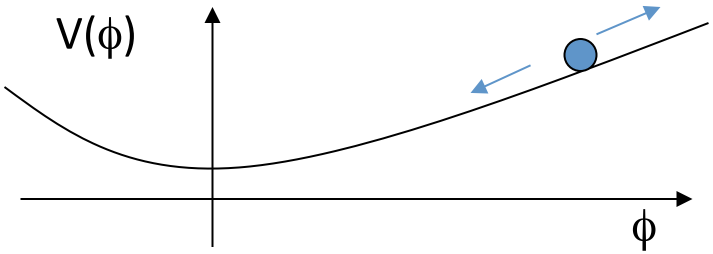
accelerated expansion
(or contraction)

radiation or matter era

decelerated expansion

Vacuum fluctuations

Hawking '82, Starobinsky '82, Guth & Pi '82



quantum vacuum

- *small-scale/underdamped zero-point fluctuations* $\delta\phi_k \approx \frac{e^{-ik\eta}}{\sqrt{2k}}$
- *large-scale/overdamped perturbations in growing mode*

  linear evolution $\Rightarrow$ *Gaussian random field*

$$P_{\delta\phi}(k = aH) \approx \frac{4\pi k^3 \left|\delta\phi_k{}^2\right|}{(2\pi)^3} = \left(\frac{H_k}{2\pi}\right)^2$$

light fields (m<3H/2) `frozen-in' on large scales

massive fields (m>3H/2) remain underdamped as k->0

gauge-invariant variables, Q and R

- field perturbation $\delta\phi$ and metric perturbation ψ are

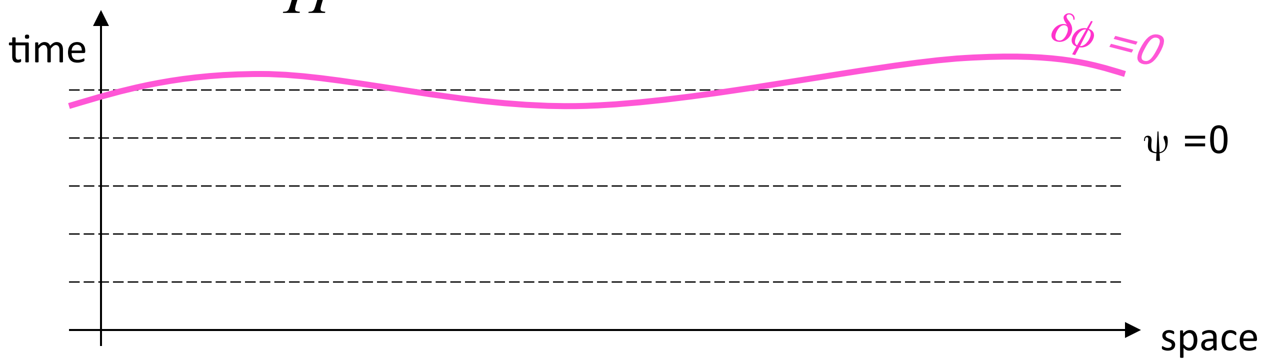
  gauge-dependent in an inhomogeneous spacetime

$$\delta\phi \rightarrow \delta\phi + \dot{\phi}\,\delta t, \quad \psi \rightarrow \psi + H\,\delta t$$

- Sasaki-Mukhanov gauge-invariant variable

$$Q = \delta\phi - \frac{\dot{\phi}}{H}\psi$$

field perturbation on flat (ψ =0) hypersurfaces

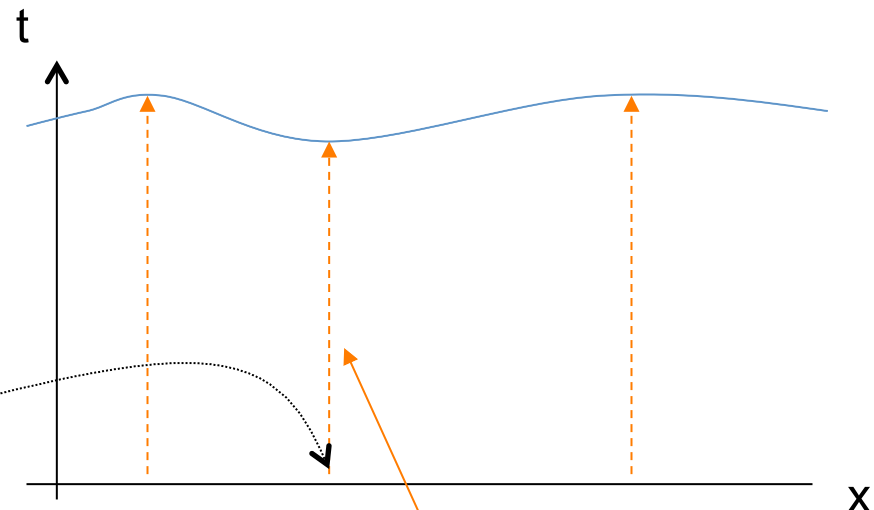


- curvature of uniform-density ($\delta\phi$ =0) hypersurfaces

$$R = \psi - \frac{H}{\dot{\phi}}\delta\phi = -\frac{H}{\dot{\phi}}Q$$

the δN formalism

in radiation-dominated era
curvature perturbation ζ on
uniform-density hypersurface



$$N = \int_{initial}^{final} H\, dt$$

during inflation field
perturbations $\phi(x,t_i)$ on initial
spatially-flat hypersurface

on large scales, neglect spatial gradients, treat as "separate universes"

the δ N formalism

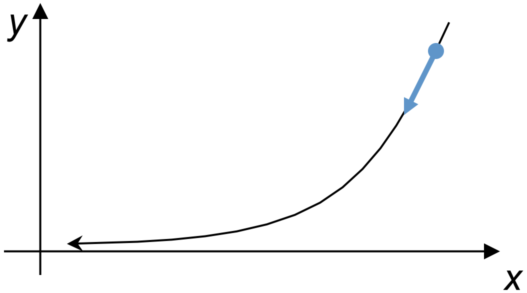
$$\zeta = N(\phi_{initial}) - \bar{N} \approx \sum_I \frac{\partial N}{\partial \phi_I} \delta\phi_I$$

Starobinsky `85; Sasaki & Stewart `96
Lyth & Rodriguez '05 – works to any order

Cosmological perturbations on large scales

- **adiabatic perturbations** *e.g.,* $$\delta\left(\frac{n_\gamma}{n_B}\right) \propto \frac{\delta n_\gamma}{n_\gamma} - \frac{\delta n_B}{n_B} = 0$$
  - *perturb along the background trajectory*

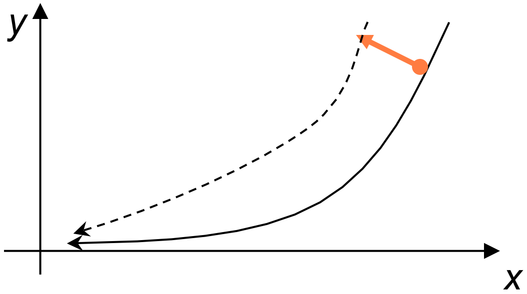
$$\frac{\delta x}{\dot{x}} = \frac{\delta y}{\dot{y}} = \delta T$$



  - *e.g, single-field perturbations along slow-roll attractor*
  - ***adiabatic perturbations stay adiabatic***

- **entropy perturbations**
  - *perturb off the background trajectory*

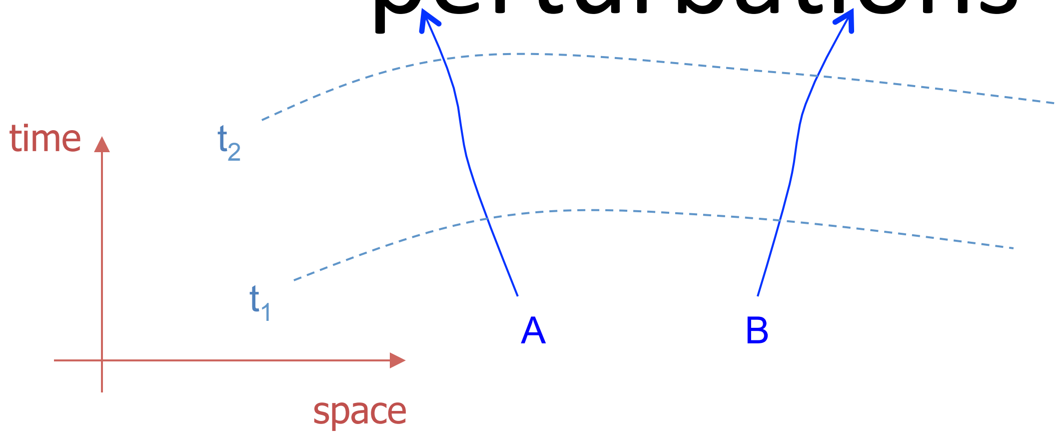
$$\frac{\delta x}{\dot{x}} \neq \frac{\delta y}{\dot{y}}$$



  - *e.g., baryon-photon **isocurvature** perturbation:*

Conserved cosmological perturbations

Lyth & Wands 2003



For every quantity, x, that obeys a **local conservation equation**

$$\frac{dx}{dN} = y(x) \quad , \qquad e.g. \quad \dot{\rho}_m = -3H\rho_m$$

where $dN = Hdt$ is the locally-defined expansion along comoving worldlines

there is a **conserved perturbation** $\zeta_x \equiv \delta N = \dfrac{\delta x}{y(x)}$

where perturbation $\delta x = x_A - x_B$ is a evaluated on hypersurfaces separated by uniform expansion $\Delta N = \Delta \ln a$

examples:

(i) total energy conservation:

$$\frac{d\rho}{dN} = H^{-1}\dot{\rho} = -3(\rho + P)$$

for perfect fluid / adiabatic perturbations, $P=P(\rho)$

$$\Rightarrow \quad \zeta_\rho = \frac{\delta\rho}{3(\rho + P)} \quad \text{conserved}$$

(ii) energy conservation for non-interacting perfect fluids:

$$H^{-1}\dot{\rho}_i = -3(\rho_i + P_i) \quad \text{where } P_i = P_i(\rho_i) \quad \Rightarrow \quad \zeta_i = \frac{\delta\rho_i}{3(\rho_i + P_i)}$$

(iii) conserved particle/quantum numbers (e.g., B, B-L,…)

$$H^{-1}\dot{n}_i = -3n_i \quad \Rightarrow \quad \zeta_i = \frac{\delta n_i}{3n_i}$$

adiabatic density perturbations from inflaton field

- *quantum fluctuations, Q, on spatially flat hypersurfaces during inflation*

$$\zeta = -R = \left(-\frac{H}{\dot{\sigma}} Q \right)_{k=aH}$$

- *produce density perturbations in radiation-dominated era*

$$\Rightarrow \quad \left\langle \frac{\delta T^2}{T^2} \right\rangle_{SW} \approx \frac{1}{25} \left\langle \zeta^2 \right\rangle \approx \frac{1}{25} \left(\frac{H^2}{2\pi\dot{\sigma}} \right)^2_{k=aH}$$

slow time-dependence during inflation -> weak scale-dependence

$$\text{tilt}: \quad n_\zeta - 1 \equiv \frac{d \ln \left\langle \zeta^2 \right\rangle}{d \ln k} \approx -6\varepsilon + 2\eta_\sigma \quad << \quad 1$$

$$\text{slow roll parameters} \quad \varepsilon = -\dot{H} / H^2 \quad , \quad \eta_\sigma = m_\sigma^2 / 3H^2$$

Primordial Density Perturbation (II)

epoch of primordial nucleosynthesis

perturbed cosmic fluid consists of

- photons, ζ_γ, neutrinos, ζ_ν, baryons, ζ_B, cold dark matter, ζ_{CDM}, (+quintessence, ζ_Q)

- total density perturbation, or curvature perturbation

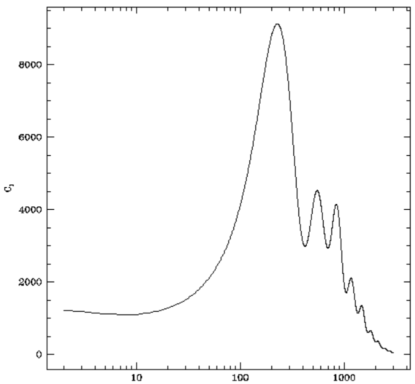
$$R = \sum_i \left(\frac{\dot{\rho}_i}{\dot{\rho}} \right) \zeta_i$$

- relative density perturbations, or isocurvature perturbtns

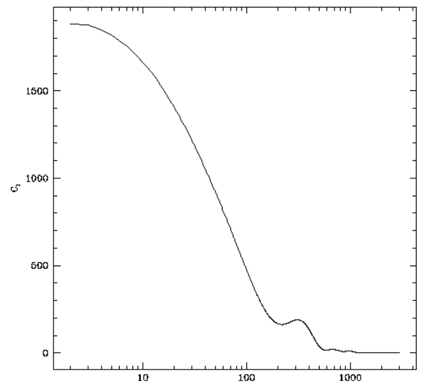
$$S_i = 3\left(\zeta_i - \zeta_\gamma \right)$$

microwave background signatures:

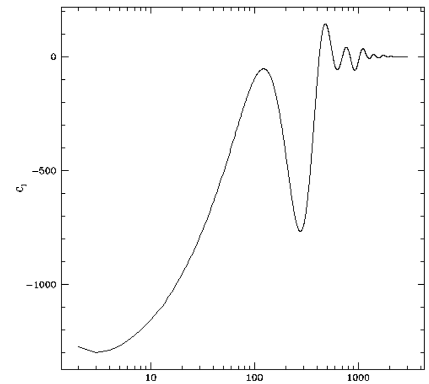
$C_l = A^2 \times \quad + B^2 \times \quad + 2\,A\,B \cos\Delta \times$



adiabatic



CDM isocurvature



correlation

Bucher, Moodley & Turok '00 } $n_S = 1$

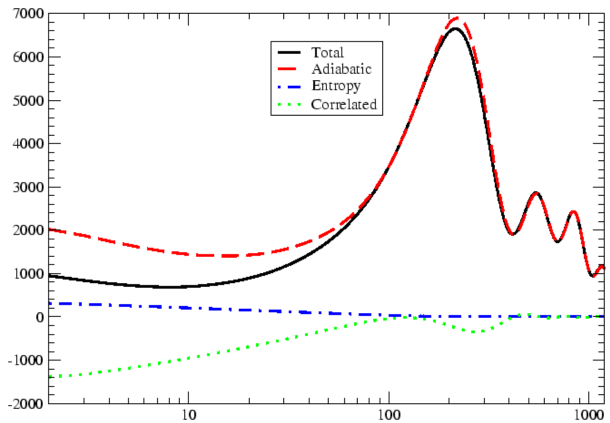
Trotta, Riazuelo & Durrer '01

Amendola, Gordon, Wands & Sasaki '01

best-fit to Boomerang, Maxima & DASI

$B/A = 0.3$, $\cos\Delta = +1$, $n_S = 0.8$

$\omega_b = 0.02$, $\omega_{cdm} = 0.1$, $\Omega_\Lambda = 0.7$



tensor metric perturbations

- *transverse, traceless metric perturbations*

$$\delta g_{ij}(t,x) \approx a^2 \int d^3k \, e_{ij}^{(+,\times)}(k) \, h_k(t) \, e^{ikx}$$

- *amplitude, $h(t)$, obeys same wave equation for massless field in an unperturbed FRW cosmology, $\delta\varphi = M_{Pl} h / (32\pi)^{1/2}$*

- *remain decoupled from matter perturbations (at first order)*

$$\Rightarrow \quad \langle T^2 \rangle \quad \approx \quad 2\left(\frac{32\pi}{M_{Pl}{}^2}\right)\left(\frac{H}{2\pi}\right)^2_{k=aH}$$

$$\text{tilt}: \quad n_T \quad \equiv \quad \frac{d\ln\langle T^2 \rangle}{d\ln k} \approx \quad -2\varepsilon \qquad \text{where} \quad \varepsilon \equiv -\dot{H}\big/H^2$$

“smoking gun” for inflation...

- *inflation predicts primordial gravitational wave background*

$$\left\langle T^2 \right\rangle \approx \left(\frac{V}{M_{Pl}{}^4} \right)_{k=aH}$$

- *could be* $$\left(\frac{10^{16}\,GeV}{M_{Pl}} \right)^4 \approx 10^{-12}$$

- *or could be* $$\left(\frac{1\,TeV}{M_{Pl}} \right)^4 \approx 10^{-64}$$

- *only detectable if inflationary scale > 10^{15} GeV*