

## Lecture 2

### Scalar perturbations

scalar field

$$\varphi(t, \vec{x}) = \varphi_0(t) + \Delta\varphi(t, \vec{x})$$

homogeneous  
background

inhomog.  
perturbation

expand order-by-order in small parameter,  $\epsilon$   
(not slow-roll  $\epsilon$ !)

$$\begin{aligned}\varphi &= \varphi_0(t) + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \delta_n \varphi(t, \vec{x}) \\ &= \varphi_0 \cancel{(\vec{x})} + \epsilon \delta_1 \varphi + \frac{1}{2} \epsilon^2 \delta_2 \varphi + \dots\end{aligned}$$

keep only terms at first-order in  $\epsilon$   
 $\Rightarrow$  "linear perturbations"

usually drop small parameter  $\epsilon$

$$\varphi = \varphi_0 + \delta\varphi + \mathcal{O}(\delta\varphi^2)$$

Fourier transform  $\rightarrow$   $k$ -space

$$\delta\varphi(t, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} \cdot \delta\varphi_{\vec{k}}(t) \cdot e^{i\vec{k} \cdot \vec{x}}$$

note that for real  $\delta\varphi \Rightarrow \delta\varphi_{\vec{k}} = \delta\varphi_{-\vec{k}}^*$

inverse

$$\delta\varphi_{\vec{k}}(t) = \int d^3x \delta\varphi(t, \vec{x}) e^{-i\vec{k} \cdot \vec{x}}$$

eigenfunctions of spatial Laplacian

$$\begin{aligned} \nabla^2(\delta\varphi_{\vec{k}}(t) \cdot e^{i\vec{k} \cdot \vec{x}}) &= -\delta^{ij} k_i k_j (\delta\varphi_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}) \\ &= -k^2 (\delta\varphi_{\vec{k}}(t) \cdot e^{i\vec{k} \cdot \vec{x}}) \end{aligned}$$

form complete orthonormal basis

$$\begin{aligned} \int \cancel{d^3x} d^3x e^{i\vec{k}_1 \cdot \vec{x}} e^{i\vec{k}_2 \cdot \vec{x}} \\ = (2\pi)^3 \cdot \delta^{(3)}(\vec{k}_1 + \vec{k}_2) \end{aligned}$$

Statistics: field = distribution = ensemble

power spectrum

$$\langle \delta C_{\vec{k}_1} \delta C_{\vec{k}_2} \rangle = (2\pi)^3 \cdot P_{\text{sc}}(k_1) \cdot \delta^3(\vec{k}_1 + \vec{k}_2)$$

in real space, variance

$$\begin{aligned} \langle C^2(\vec{x}) \rangle &= \int \frac{d^3 \vec{k}_1 d^3 \vec{k}_2}{(2\pi)^6} \langle C_{\vec{k}_1} C_{\vec{k}_2} \rangle e^{i(\vec{k}_1^2 + \vec{k}_2^2) \cdot \vec{x}} \\ &= \int \frac{d^3 \vec{k}_1 d^3 \vec{k}_2}{(2\pi)^6} \cdot (2\pi)^3 P_{\text{sc}}(k_1) \delta^3(\vec{k}_1 + \vec{k}_2) \\ &\quad \times e^{i(\vec{k}_1^2 + \vec{k}_2^2) \vec{x}^2} \\ &= \int \frac{d^3 \vec{k}_1}{(2\pi)^3} P_{\text{sc}}(k_1) \\ &= \int \frac{4\pi k_1^3 P_{\text{sc}}(k_1)}{(2\pi)^3} \frac{dk_1}{k_1} \\ &= \int P_{\text{sc}}(k_1) dk_1 \end{aligned}$$

dimensionless power per  $\ln k$

$$P_{\text{sc}}(k_1) = \frac{4\pi k_1^3}{(2\pi)^3} P_{\text{sc}}(k)$$

bispectrum

$$\begin{aligned} \langle \delta C_{\vec{k}_1} \delta C_{\vec{k}_2} \delta C_{\vec{k}_3} \rangle &= (2\pi)^3 B_{\text{sc}}(k_1, k_2) \\ &\quad \times \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \end{aligned}$$

$$= 0 \quad \text{for Gaussian perturbations} \\ (\text{and for all odd moments, } \langle Q^5 \rangle = 0)$$

if we take all first-order perturbations to be Gaussian  
 (free fluctuations of linear wave equations)  
 then higher-order perturbations are non-Gaussian

Vector perturbations :  
decompose

$$V_i = \partial_i V^{(s)} + V_i^{(v)}$$

↑ potential flow  
(longitudinal)  
"scalar"

$$\vec{V} = \vec{\nabla} V^{(s)} + \vec{V}^{(v)}$$

↑ divergence-free  
(transverse)  
"vector"

where

$$\frac{\partial}{\partial x^i} (V_i^{(v)}) = 0$$

Fourier basis

$$V^{(s)}(t, \vec{x}) = \int \frac{d^3 \vec{k}}{(2\pi)^3} V_{\vec{k}}^{(s)}(t) e^{i \vec{k} \cdot \vec{x}}$$

$$\Rightarrow \partial_i V^{(s)} = i k_i V^{(s)}$$

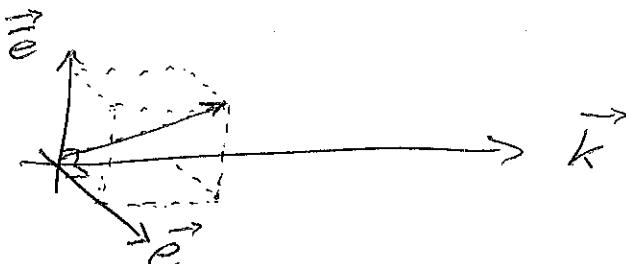
$$V_i^{(v)} = \int \frac{d^3 \vec{k}}{(2\pi)^3} \left\{ V_{\vec{k}}^{(v)}(t) \vec{e}_i(\vec{k}) + \bar{V}_{\vec{k}}^{(v)} \bar{\vec{e}}_i(\vec{k}) \right\} e^{i \vec{k} \cdot \vec{x}}$$

where  $\vec{e}_i$  &  $\bar{\vec{e}}_i$  are orthonormal

$$\delta^{ij} \vec{e}_i \cdot \vec{e}_j = \delta^{ij} \bar{\vec{e}}_i \cdot \bar{\vec{e}}_j = 1$$

$$\delta_{ij} \vec{e}_i \cdot \bar{\vec{e}}_j = 0$$

and transverse to  $\vec{k}$  :  $\delta^{ij} \vec{e}_i k_j = \delta^{ij} \bar{\vec{e}}_i k_j = 0$



## Tensor perturbations

decompose

$$T_{ij} = \cancel{d_i d_j S} + \frac{1}{2} (\partial_i V_j + \partial_j V_i) + h_{ij}$$

where  $V_i$  is transverse

$$h_{ij} \text{ is transverse } \delta^{ij} \partial_i h_{jk} = 0$$

$$\text{and trace-free } \delta^{ij} h_{ij} = 0$$

## Fourier basis

$$h_{ij}(t, \vec{x}) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \left\{ h_{\vec{k}} q_{ij}^{(+)}(\vec{k}) + \bar{h}_{\vec{k}} q_{ij}^{(x)}(\vec{k}) \right\} e^{i \vec{k} \cdot \vec{x}}$$

where

$$q_{ij}^{(+)} = \frac{1}{\sqrt{2}} (e_i \bar{e}_j - \bar{e}_i e_j)$$

$$q_{ij}^{(x)} = \frac{1}{\sqrt{2}} (e_i \bar{e}_j + \bar{e}_i e_j)$$

these are orthonormal, transverse and tracefree

(given that  $e_i$  &  $\bar{e}_i$  are orthonormal & transverse).

$$q_{ij}^{(x)} q_{kl}^{(+)i} = q_{ij}^{(+)} q_{kl}^{(+)} = 1$$

$$q_{ij}^{(x)} k^i = q_{ij}^{(+)} k^i = q_{ij}^{(+)} q_{ij}^{(x)} = 0$$

$$\delta^{ij} q_{ij}^{(+)} = \delta^{ij} q_{ij}^{(x)} = 0$$

## Metric perturbations

FRW

homogeneous background:

$$\begin{aligned}
 \text{FRW metric } ds^2 &= \cancel{\alpha^2 dt^2} \\
 &= -dt^2 + \alpha^2 \cancel{\delta_{ij} dx^i dx^j} \\
 &\quad \text{(scale factor)} \\
 &= \alpha^2 [-d\eta^2 + \delta_{ij} dx^i dx^j] \\
 &\quad \text{(conformal time)} \\
 \eta &= \int dt / \alpha
 \end{aligned}$$

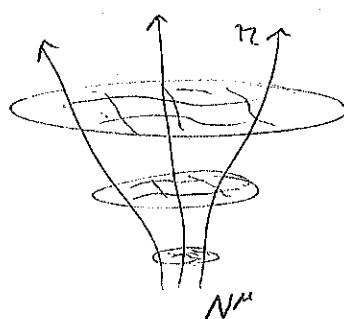
inhomogeneous pertbs:  $ds^2 = \alpha^2 [-(1+2A)d\eta^2 + B_{,i} d\eta dx^i + \underbrace{(C_{ij})}_{B_{,i}} dx^i dx^j]$

$$\begin{aligned}
 ds^2 &= \alpha^2 \left[ -(1+2A) d\eta^2 + \underbrace{(B_{,i} - S_i)}_{B_{,i}} dt dx^i \right. \\
 &\quad \left. + \underbrace{[(1-2\psi)\delta_{ij} + 2E_{,ij} + 2F_{(i,j)} + h_{ij}]}_{C_{ij}} dx^i dx^j \right]
 \end{aligned}$$

scalar pertbs:  $A, B, \psi, E = 4 \times 1 \text{ d.o.f.}$

vector pertbs:  $S_i, F_i \quad \text{transverse } 2 \times 2 \text{ "}$

tensor pertbs:  $h_{ij} \quad \text{transverse, tracefree } 2 \times \frac{1}{2} \text{ "}$   
 symmetric metric:  $\frac{1}{2} 10 \text{ d.o.f.}$



hypersurface orthogonal 4-vector field

$$N^\mu = \frac{dx^\mu}{d\tau}$$

$$N_\mu \propto \frac{dn}{dx^\mu} \text{ trans}$$

$$N^\mu = \frac{1}{a} (1 - \phi, -B_i) \quad \text{to first order}$$

$$N_\mu = -a (1 + \phi, 0)$$

$$\nabla_\nu N_\mu = -\frac{1}{3} \theta P_{\mu\nu} + \partial_{\mu\nu} + \cancel{\partial}_{\mu\nu}^0 - a_\mu N_\nu$$

$$P_{\mu\nu} = g_{\mu\nu} + N_\mu N_\nu$$

$$\text{expansion : } \theta = \frac{3}{a} \left( \frac{a'}{a} (1 - \phi) - \psi' + \frac{1}{3} \delta^2 E \right)$$

$$\text{shear : } \gamma_{ij} = (\delta_i \delta_j - \frac{1}{3} \delta_{ij} \delta^2) \beta + \text{vector} + h_{ij}$$

$$\text{roticity : } \beta = E' - B$$

$$\text{acceleration : } a_i = \dot{\phi}_i$$

$$\text{intrinsic curvature : } {}^{(3)}R = \frac{4}{a^2} \delta^2 \psi$$

$\psi$  = "curvature perturbation"  
dimensionless.

## LECTURE 3

### Gauge transforms:

homogeneous background has preferred coord ~~system~~ "t"  
 $\phi = \phi_0(t)$

coord system arbitrary in inhomog. cosmology

at point P  $\phi_p = \phi_0(t) + \tilde{\epsilon} \delta_\phi \phi(t, x) + \mathcal{O}(\tilde{\epsilon}^2)$

perform change coords at physical point P (passive transform)

let  $t \rightarrow \tilde{t} = t + \tilde{\epsilon} \delta t + \mathcal{O}(\tilde{\epsilon}^2)$

$$\cancel{\partial_i \phi} \cancel{\partial^i} = \cancel{x^i} + \tilde{\epsilon} \delta x^i + \mathcal{O}(\tilde{\epsilon}^2)$$

$$x^i = \cancel{x^i} + \tilde{\epsilon} (\delta x^i_0 + \delta x^i) + \dots$$

scalar      vector

$$\phi_p = \phi_0(\tilde{t}) + \tilde{\epsilon} \delta_\phi \phi(\tilde{t}, \tilde{x}) + \mathcal{O}(\tilde{\epsilon}^2)$$

$$= \phi_0(t) + \tilde{\epsilon} \dot{\phi}_0(t) \cdot \delta t$$

$$+ \tilde{\epsilon} \delta_\phi \phi(t, x) + \mathcal{O}(\tilde{\epsilon}^2)$$

$$= \phi_0(t) + \tilde{\epsilon} \delta_\phi \phi(t, x)$$

$$\Rightarrow \boxed{\tilde{\delta}_1 \phi = \delta_1 \phi - \dot{\phi}_0 \cdot \delta t}$$

gauge transform for scalars

e.g.  $\tilde{\delta} \rho = \delta \rho - \dot{\rho} \delta t$  density

$$\tilde{\delta} p = \delta p - \dot{p} \delta t$$
 pressure

note:  $\tilde{v} = v + \delta v'$  velocity

gauge-invariant combination

$$\delta P_{\text{rad}} = \delta P - \left( \frac{\dot{p}}{\dot{\rho}} \right) \delta \rho$$

$$= \delta P - c_s^2 \delta \rho \quad \text{where } c_s^2 = \dot{p}/\dot{\rho}$$

$$= 0 \quad \text{for adiabatic perturb.}$$

physical interpret.:  $\delta P_{\text{rad}} = \delta P$  / coord system where  $\delta \rho = 0$   
"uniform-density gauge"

## Gauge transform of metric perturbations

$$ds^2 = \tilde{a}^2 \left[ - (1 + 2A) d\eta^2 + 2B_i dx^i + (\delta_{ij} + 2C_{ij}) dx^i dx^j \right] \quad ?$$

$$\begin{aligned} \eta &\rightarrow \tilde{\eta} + \delta\eta = \tilde{\eta} \\ x^i &\rightarrow \tilde{x}^i + \delta x^i = \tilde{x}^i \\ x^i &+ (\delta x^i + \bar{\delta}x^i) \end{aligned}$$

scalar + vector

$$\text{scalars } A \rightarrow A - \frac{a'}{a} \delta\eta - \delta\eta' \quad \frac{a'}{a} = H = aH$$

$$\psi \rightarrow \psi + \frac{a'}{a} \delta\eta$$

$$B \rightarrow B + \delta\eta - \delta x' \quad \} \quad \beta = E - B$$

$$E \rightarrow E - \delta x \quad \rightarrow \tilde{\delta} = \beta - \delta\eta \quad \text{scalar shear}$$

$$\begin{aligned} S_i &\rightarrow S_i + \bar{\delta}x'_i \quad \} \quad S_i + F_i \text{ is gauge inv.} \\ F_i &\rightarrow F_i - \bar{\delta}x_i \quad \} \quad \text{vector shear} \end{aligned}$$

$$h_{ij} \rightarrow \tilde{h}_{ij} = h_{ij} \quad \begin{aligned} &\text{automatically gauge inv} \\ &\equiv \text{gauge independent} \\ &\text{(at 1st order!)} \end{aligned}$$

tensor shear  
= "gravitational waves"