Lecture 3

constructing gauge-invariant variables

pick reference frame (coords) to completely fix gauge

e.g. <u>longitudinal</u> $\tilde{E} = \tilde{B} = 0$, $\tilde{S}_i = 0$

(note that the 4-vector field is then zero-shear, $\tilde{\sigma} = 0$)

$$\tilde{E} = 0 \quad \Rightarrow \quad \delta x = E \quad \text{from arbitrary gauge}$$
$$\tilde{B} = 0 \quad \Rightarrow \quad \delta\eta = -B + \delta x' = E' - B$$
$$\tilde{S}_i = 0 \quad \Rightarrow \quad \delta x^{i\,\prime} = -S^i$$

remaining variables are gauge invariant

$$\tilde{\psi} = \psi + \frac{a'}{a}\delta\eta = \boxed{\psi + \frac{a'}{a}(E' - B) \equiv \Psi}$$ Mukhanov et al

$$\tilde{A} = A - \frac{a'}{a}\delta\eta - \delta\eta' = \boxed{A - \frac{a'}{a}(E' - B) - (E' - B)' = \Phi}$$

$$\widetilde{\delta\rho} = \delta\rho - \rho'(E' - B) \,,$$

$$\tilde{v} = v + E'$$

$$\tilde{F}_i = F_i - \delta x_i = F_i - \int S_i \, d\eta$$

gauge-invariant vector metric pertbn :

$$\Sigma_i = S_i + F_i'$$

comoving + orthogonal

$$v^i = \frac{dx^i}{d\eta}$$

$$v^i \longrightarrow \tilde{v}^i = v^i + \xi^i$$

scalar $\quad v = v + \delta x$

vector $\quad v^{(v)i} = v^{(v)i} + \delta x^i$

comoving - threading $\quad \tilde{V}^i = 0 \Rightarrow \delta x^i = -v^i$

orthogonal - slicing $\quad \tilde{B} = 0 \Rightarrow \delta\eta = -B + \delta x'$

$$= -(v + B)$$

curvature pertbn : $\tilde{\psi} = \psi + \frac{a'}{a}\delta\eta$

$$= \boxed{\mathcal{R} = \psi - \frac{a'}{a}(v+B)}$$

density $\quad \boxed{\tilde{\delta\rho}_{con} = \delta\rho + \rho_0'(v + B)}$

$$= \delta\rho - 3\frac{a'}{a}(\rho + P)(v + B)$$

note : scalar field momentum

$$(\rho + P)(v + B) = -\dot{\phi}\,\delta\varphi$$

$$\Rightarrow \quad v + B = -\frac{\delta\varphi}{\dot{\varphi}}$$

$\Rightarrow$ comoving + orthogonal $\Rightarrow$ uniform-field uniform-density

$$\widetilde{\delta\rho} = 0 \quad \Rightarrow \quad \delta\eta = \frac{\delta\rho}{\rho_0'}$$

$$-\tilde{\psi} = \zeta = -\psi - \frac{a'}{a}\frac{\delta\rho}{\rho_0'}$$

$$\boxed{\zeta = -\psi - \frac{H}{\dot\rho}\,\delta\rho}$$

almost same as $\mathcal{R}$ on large scales

$$\zeta + \mathcal{R} = -\frac{H}{\dot\rho}\left(\delta\rho + \rho_0'(v+B)\right)$$

$$\boxed{\zeta + \mathcal{R} = -\frac{H}{\dot\rho}\,\widetilde{\delta\rho}_{com}}$$

$= 0$ in single-field, slow-roll inflation

multi-fluids: $\rho = \sum_I \rho_I$

$$\zeta = -\psi - H\,\frac{\sum \delta\rho_I}{\dot\rho}$$

$$\boxed{\zeta = \sum_I \left(\frac{\dot\rho_I}{\dot\rho}\right)\zeta_I}$$

where $\zeta_I = -\psi - \frac{H}{\dot\rho_I}\,\delta\rho_I$

relative / entropy perturbation (w.r.t. photons)

$$\boxed{S_I = 3(\zeta_I - \zeta_\gamma)}$$

$$= 3H\left(\frac{\delta\rho_\gamma}{\dot\rho_\gamma} - \frac{\delta\rho_I}{\dot\rho_I}\right)$$

e.g baryons: $S_B = \frac{\delta\rho_B}{\rho_B} - \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} = \frac{\delta n_B}{n_B} - \frac{\delta n_\gamma}{n_\gamma}$

spatially - flat gauge

$$\tilde{\psi} = \tilde{E} = 0$$

$$\Rightarrow \quad \delta\eta = -\frac{a}{a'}\psi, \qquad \delta x = E$$

density: $\widetilde{\delta\rho} = \delta\rho - \frac{\dot{\rho}}{H}\psi$

$$= + \frac{\dot{\rho}}{H}\zeta$$

scalar-field $\widetilde{\delta\varphi} = \delta\varphi - \frac{\dot{\varphi}}{H}\psi$

$$= -\frac{\dot{\varphi}}{H}\mathcal{R} = Q$$

First-order field equations

Einstein equations:

Two constraint equations:

$$
\begin{aligned}
3h(\psi' + h\phi) + (k^2 - 3\mathcal{K})\psi + hk^2\sigma &= -4\pi G a^2 \delta\rho && \text{(energy density)} \\
\psi' + h\phi + \mathcal{K}\sigma &= -4\pi G a^2(\rho + p)(v + B) && \text{(momentum)}
\end{aligned}
$$

Two evolution equations:

$$
\begin{aligned}
\psi'' + 2h\psi' - \mathcal{K}\psi + h\phi' + (2h' + h^2)\phi &= 4\pi G a^2 \left(\delta p - \frac{2}{3}k^2 a^2 \Pi\right) && \text{(curvature)} \\
\sigma' + 2h\sigma - \phi + \psi &= 8\pi G a^2 \Pi && \text{(shear)}
\end{aligned}
$$

Energy-momentum conservation:

$$
\begin{aligned}
\delta\rho'_\alpha + 3h(\delta\rho_\alpha + \delta p_\alpha) &= (\rho_\alpha + p_\alpha)\left[k^2(v_\alpha + E') + 3\psi'\right] && \text{(energy density)} \\
[(\rho_\alpha + p_\alpha)(v_\alpha + B)]' &= -(\rho_\alpha + p_\alpha)\left[4h(v_\alpha + B) + \phi\right] - \delta p_\alpha + \frac{2}{3}(k^2 - 3\mathcal{K})\Pi_\alpha
\end{aligned}
$$

(momentum)

field equations:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \longrightarrow \begin{array}{ll} \text{energy constraint} & G_{00} \\ \text{mtm constraint} & G_{0i} \\ \text{evolution equations} & G_{ij} \end{array}$$

$$\Downarrow$$

$$\nabla^\mu T_{\mu\nu} = 0 \longrightarrow \begin{array}{ll} \text{energy conservation} & \nabla^\mu T_{\mu 0} \\ \text{mtm. conservation} & \nabla^\mu T_{\mu i} \end{array}$$

e.g. – energy conservation in uniform-density gauge

$$\Rightarrow \zeta' = -\mathcal{H} \frac{\delta P_{nad}}{\rho + P} - \frac{1}{3}\nabla^2 (v + E')$$

non-adiabatic pressure

comoving shear $\to 0$ on large scales

$\Rightarrow \zeta =$ constant for ad. pertbns on large-scales

– energy + mtm constraints

$$\Rightarrow \nabla^2 \Psi = 4\pi G a^2 \delta\rho_{com}$$

$$= \frac{3}{2} a^2 H^2 \frac{\delta\rho_{com}}{\rho_{com}}$$

$\delta = \frac{2}{3} \frac{\nabla^2 \Phi}{a^2 H^2}$

$\propto a$ in matter era

$\Rightarrow \delta\rho_{com}/\rho_{com} \longrightarrow 0$ on large scales

and $\zeta = -\mathcal{R} + \frac{1}{3} \frac{\delta\rho_{com}}{\rho + P}$

$\Rightarrow \zeta \longrightarrow -\mathcal{R}$ on large scales

– energy constraint in long. gauge

$$\Rightarrow \zeta = -\Psi + \frac{\mathcal{H}(\Psi' + \mathcal{H}\Phi) - (\nabla^2 \Psi/3)}{\mathcal{H}' - \mathcal{H}^2}$$

- evolution equations in long. gauge ($B=0$)
  $\Rightarrow \Psi = \Phi$ ~~for~~ in absence of anisotropic pressure [?]
  $\Pi = 0$
  e.g. ~~m~~ minimally-coupled scalar fields

- ~~energy~~ mtm constraint in long. gauge

$$\mathcal{R} = \Psi - \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2}\left(\Psi' + \mathcal{H}\Phi\right)$$

$\Rightarrow$ for growing mode on large scales $\Psi' = \cancel{\text{const}}$

$$\mathcal{R} \simeq \left(\frac{\mathcal{H}' - 2\mathcal{H}^2}{\mathcal{H}' - \mathcal{H}^2}\right)\Phi \quad \text{if } \Psi = \Phi$$

$$\simeq \frac{5+3w}{3(1+w)}\Phi \quad \text{for } \frac{P}{\rho} = w$$

$$\simeq -\zeta$$

$\Phi \simeq -\frac{3}{5}\zeta$ ~~$\frac{5}{3}\Phi$~~ for $w=0$, matter

$\Phi \simeq -\frac{2}{3}\zeta$ ~~$\frac{3}{2}\Phi$~~ for $w=\frac{1}{3}$, radiation

- mtm constraint in comoving ~~flow~~ gauge

$$\mathcal{R}' = -\mathcal{H}A$$

mtm conservation

$$0 = A + \frac{\delta p}{\rho + p} + \frac{2}{3}\frac{\nabla^2\Pi}{\rho + P}$$

pressureless matter ($\delta p = 0$): $A = 0$ "synchronous"
$\mathcal{R}' = 0$ on all scales

even in presence of other fluids (e.g. radiation)
there exists synchronous gauge comoving with pressureless CDM.
~~...~~
(but $\mathcal{R}' \neq 0$ in general)