Seeking Gold in Sand

Financial applications of Random Matrix Theory in stock market data

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Preview of the talk

1. What is Random Matrix Theory?
2. Data Overview
3. Random Matrix Theory: The Marčenko-Pastur Law
4. Mode & Clustering Analyses
5. A Multi-layer Structured Correlation Model & Its Predictions
6. Effect of Layer Division & an Excellent Match
7. Summary & Further Developments
8. Q&A Time

1 What is Random Matrix Theory?

Random Matrix Theory (RMT) is the study of *matrices with random variable (r.v.) entries* , e.g.

$$\begin{bmatrix} X_{11} & X_{12} & \cdots \\ X_{21} & X_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

In particular, it concerns the *emergent behaviours* of random matrices in the asymptotic limit.

Introduced by Wishart (1928), RMT gained prominence when Wigner (1950s) applied the theory to spacing of energy levels in nuclear physics.

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2 Data overview: S&P 500

Procedure

Price indices $\rightarrow$ logarithmic returns $\rightarrow$ de-meaned and normalised data.

$$\begin{array}{cc} & P \text{ stocks} \\ T \text{ days} & \underbrace{\begin{bmatrix} 126.8 & 30.5 & \cdots \\ 126.3 & 30.7 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\text{raw data}} \end{array} \mapsto \underbrace{\begin{bmatrix} -0.9 & 1.7 & \cdots \\ 1.5 & 0.3 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\text{standardised data } X} \mapsto \underbrace{\begin{bmatrix} 1.0 & 0.2 & \cdots \\ 0.2 & 1.0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\text{correlation matrix } E}$$

The log return is

$$R_i = \log \frac{p_i}{p_{i-1}} (\approx \frac{p_i - p_{i-1}}{p_{i-1}}), \quad i > 1$$

where p_i is the i-th trading day price index. The empirical correlation matrix is

$$E = \frac{1}{T} X^t X.$$

3 Random Matrix Theory: the Marčenko-Pastur law

Covariance-correlation matrices are of fundamental importance to modern portfolio theory.

They belong a class of random matrices called the Wishart ensemble.

An important universality law for this ensemble in RMT:

The Marčenko-Pastur law

If $X : T \times P$ has independently identically distributed (i.i.d.) r.v. entries with mean 0 and variance 1, then the limiting eigenvalue density function (e.d.f.) of matrix $E = T^{-1}X^tX$ is

$$f(\lambda) = \frac{1}{2\pi}\frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{r\lambda}$$

as $P, T \to \infty$ and $P/T \to r \in (0, 1)$, where $\lambda_\pm = (1 \pm \sqrt{r})^2$.

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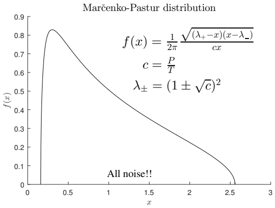
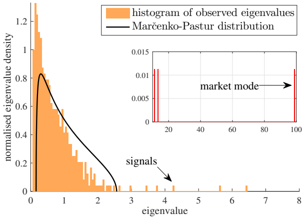
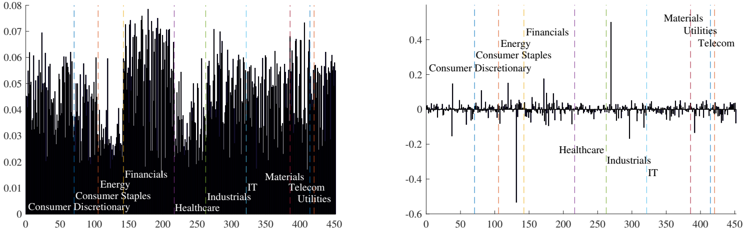
3 The Marčenko-Pastur law: a crude prediction

underlying correlation matrix C		predicted e.d.f. of E
I_P	$\Rightarrow$	
?	$\Rightarrow$	

4 Mode analysis



The market mode (L: IPR 3.98×10^{-5})
and
the lowest mode (R: IPR 0.149).

Localisation

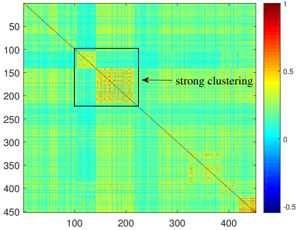
The inverse participation ratio is defined by $\mathrm{IPR}(\mathbf{v}) = \sum_{i=1}^{P} |\tilde{v}_i|^4$ where $\tilde{\mathbf{v}}$ is the vector $\mathbf{v}$ demeaned and normalised.

4 Clustering analysis

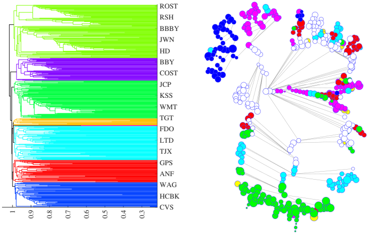
Market mode removal: $E' = E - \lambda_1 \mathbf{v}_1 \mathbf{v}_1^t$ ($\lambda_1, \mathbf{v}_1$ largest eigenvalue pair);

Dissimilarity distance: $d_{ij} = 1 - \mathrm{corr}(i, j)$;

Average linkage: $D_{IJ} = \frac{1}{|I||J|} \sum_{i \in I, j \in J} d_{ij}$.

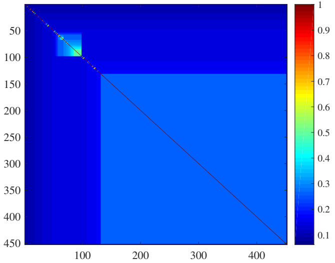


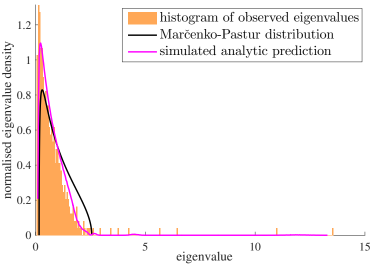
Heatmap



Dendrogram (L)
Minimum spanning tree (R)

5 A multi-layered correlation model: preview visualisation





Heatmap (L) and analytic prediction (R)
for the empirical e.d.f of a multi-layered model.

5 A multi-layered correlation model: analytic prediction

Model: $\nu(\lambda) = P^{-1}\sum_{i=1}^{P}\delta(\lambda-\lambda_i)$, the e.d.f. of underlying correlation matrix C. **Prediction**: $f(\lambda)$, the limiting e.d.f. of empirical correlation matrix E.

The Stieltjes transform pair

$$G(z) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{f(\lambda)}{\lambda - z}, \qquad f(\lambda) = \lim_{\epsilon\to 0} \mathrm{Im}\{G(\lambda + i\epsilon)\}$$

The Marčenko-Pastur equation

$$-\frac{1}{G(z)} = z - r\int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\lambda\nu(\lambda)}{1+\lambda G(z)}$$

A polynomial equation

$$[1+zG(z)]\prod_{i=1}^{P}[1+\lambda_i G(z)] = \frac{1}{r}\sum_{i=1}^{P}\lambda_i G(z)\prod_{j\neq i}^{P}[1+\lambda_j G(z)].$$

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6 The layer division process: fundamental structures

Layer division:

$$M := M_m(1, \alpha) \quad \text{a single cluster}$$

$$\Downarrow$$

$$M' := \begin{bmatrix} M_{m_1}(1, \alpha_1) & B \\ B^t & M_{m_2}(1, \alpha_2) \end{bmatrix} \quad \text{two smaller clusters}$$

where $m = m_1 + m_2$.

Fundamental structures:

$$M_n(x, y) \equiv \underbrace{\begin{bmatrix} x & y & \cdots & y \\ y & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & y \\ y & \cdots & y & x \end{bmatrix}}_{n}, \quad B = \beta\underbrace{(1, \ldots, 1)}_{m_1}^t \underbrace{(1, 1, \ldots, 1)}_{m_2}.$$

Useful techniques and results:

- elementary operations;
- the identity

$$\begin{pmatrix} S & T \\ U & V \end{pmatrix} \begin{pmatrix} I & 0 \\ -V^{-1}U & I \end{pmatrix} = \begin{pmatrix} S - TV^{-1}U & T \\ 0 & V \end{pmatrix}$$

  where V is invertible;

- the *Sherman-Morrison formula*:

$$(A + \mathbf{uv}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{uv}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}}$$

  where A is invertible and $1 + \mathbf{v}^T A^{-1}\mathbf{u} \neq 0$.

6 The layer division process: eigenvalue splitting

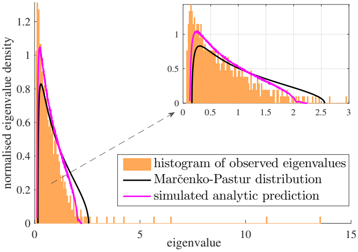
Findings:

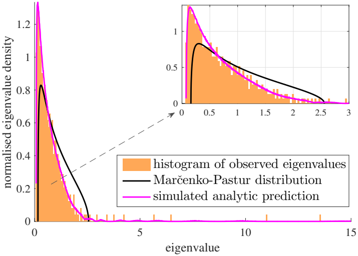
- M has an eigenvalue $\lambda_1 = 1 - \alpha$ of multiplicity $m - 1$ and an eigenvalue $\lambda_2 = 1 + (m-1)\alpha$ of multiplicity 1.
- M' has eigenvalues $1 - \alpha_{1,2}$ of multiplicity $m_{1,2} - 1$, and the remaining eigenvalues $\lambda'_{1,2}$ are roots of a quadratic polynomial with

$$\lambda'_1 + \lambda'_2 = \lambda_1 + \lambda_2.$$

The interaction correlation β perturbs these two eigenvalues, causing them to separate and repel.

6 The layer division process: an excellent match





Comparison of the predicted empirical spectral density functions of a 10-layer correlation model (L) and a 148-layer one (R) with the market mode.

7 Summary & further developments

Summary.

More considerations:

- edge asymptotics with the *Tracy-Widom law*;
- time evolution;
- fine-tuning;
- ...

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Q&A

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More information about this project at
`http://people.ds.cam.ac.uk/sw664/SUROP%20Project%202016/SUROP.html`